Phase Transition in a Log-normal Interest Rate Model

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Outline

- Introduction to interest rate modeling
- Black-Derman-Toy model
  - Generalization with continuous state variable
  - Binomial tree
- BDT model with log-normal short rate in the terminal measure
  - Analytical solution
  - Surprising large volatility behaviour
- Phase transition
- Summary and conclusions
A One-Factor Model of Interest Rates  

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Markov-functional interest rate models  
*Finance and Stochastics*, 4, 391-408 (2000)

Efficient methods for valuing interest rate derivatives  
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[4] D. Pirjol,  
Phase transition in a log-normal Markov functional model  

[5] D. Pirjol,  
Nonanalytic behaviour in a log-normal Markov functional model,  
Interest rates

Interest rates are a measure of the time value of money: what is the value today of $1 paid in the future?

![USD Zero Curve](image)

**Example**

Zero curve $R(t)$ for USD as of 27-Sep-2011. Gives the discount curve as $D(t) = \exp(-R(t)t)$. 
Motivation
Black, Derman, Toy model
Model with log-normal rates in the terminal measure
Phase Transition

Interest rates evolve in time

Example
The range of movement for the USD zero curve $R(t)$ between 22-Jun-2011 and 27-Sep-2011.
Yield curve inversion

Example
Between 2006 and 2008 the USD yield curve was ‘inverted’ - rates for 2 years were lower than the rates for shorter maturities
Interest rate modeling

- The need to hedge against movements of the interest rates contributed to the creation of interest rate derivatives.
- Corporations, banks, hedge funds can now enter into many types of contracts aiming to mitigate or exploit/leverage the effects of the interest rate movements.
- Well-developed market. Daily turnover for¹:
  - interest rate swaps $295bn
  - forward rate agreements $250bn
  - interest rate options (caps/floors, swaptions) $70bn
- This requires a very good understanding of the dynamics of the interest rates markets: interest rate models

¹The FX and IR Derivatives Markets: Turnover in the US, 2010, Federal Reserve Bank of New York
Consider a model for interest rates defined on a set of discrete dates (tenor)

\[ 0 = t_0 < t_1 < t_2 \cdots t_{n-1} < t_n \]

At each time point we have a yield curve. Zero coupon bonds \( P_{i,j} \): price of bond paying $1 at time \( t_j \), as of time \( t_i \). \( P_{i,j} \) is known only at time \( t_i \).

\( L_i = \text{Libor rate set at time } t_i \text{ for the period } (t_i, t_{i+1}) \)

\[ L_i = \frac{1}{\tau} \left( \frac{1}{P_{i,i+1}} - 1 \right) \]
Simplest interest rate derivatives: *caplets and floorlets*

Caplet on the Libor $L_i$ with strike $K$ pays at time $t_{i+1}$ the amount

\[
\text{Pay} = \max(L_i(t_i) - K, 0)
\]

Similar to a call option on the Libor $L_i$

Caplet prices are parameterized in terms of caplet volatilities $\sigma_i$ via the Black caplet formula

\[
\text{Caplet}(K) = P_{0,i+1} C_{BS}(L_i^{fwd}, K, \sigma_i, t_i)
\]

Analogous to the Black-Scholes formula.
An analogy with equities

Define the forward Libor $L_i(t)$ at time $t$, not necessarily equal to $t_i$

$$L_i(t) = \frac{1}{\tau} \left( \frac{P_{t,i}}{P_{t,i+1}} - 1 \right)$$

This is a stochastic variable similar to a stock price, and its evolution can be modeled in analogy with an equity.

Assuming that $L_i(t_i)$ is log-normally distributed one recovers the Black caplet pricing formula.

Generally the caplet price is the convolution of the payoff with $\Phi(L)$, the Libor probability distribution function

$$Caplet(K) = \int_0^\infty dL \Phi(L)(L - K)_+$$
Caplet volatility - term structure

Volatility hump at the short end
ATM caplet volatility - term structure

Actual 3m USD Libor caplet yield (log-normal) volatilities as of 27-Sep-2011
Caplet volatility - smile shape

\[ \sigma(K) \text{ vs. Strike} \]

\[ \sigma_{\text{ATM}} \]

\[ \text{Fwd} \]
Construct an interest rate model compatible with a given yield curve $P_{0,i}$ and caplet volatilities $\sigma_i(K)$

1. *Short rate models.* Model the distribution of the short rates $L_i(t_i)$ at the setting time $t_i$.
   - Black, Derman, Toy model
   - Hull, White model - equivalent with the Linear Gaussian Model (LGM)
   - Markov functional model

2. *Market models.* Describe the evolution of individual forward Libors $L_i(t)$
   - Libor Market Model, or the BGM model.
The natural (forward) measure

Each Libor $L_i$ has a different “natural” measure $\mathbb{P}_{i+1}$

Numeraire = $P_{t,i+1}$, the zero coupon bond maturing at time $t_{i+1}$

The forward Libor $L_i(t)$

$$L_i(t) = \frac{1}{\tau} \left( \frac{P_{t,i}}{P_{t,i+1}} - 1 \right)$$

is a martingale in the $\mathbb{P}_{i+1}$ measure

$$L_i(0) = L_i^{\text{fwd}} = \mathbb{E}[L_i(t_i)]$$

This is the analog for interest rates of the risk-neutral measure for equities
Simple Libor market model

Simplest model for the forward Libor $L_i(t)$ which is compatible with a given yield curve $P_{0,i}$ and log-normal caplet volatilities $\sigma_i$

Log-normal diffusion for the forward Libor $L_i(t)$: each $L_i(t)$ driven by its own separate Brownian motion $W_i(t)$

$$dL_i(t) = L_i(t)\sigma_i dW_i(t)$$

with initial condition $L_i(0) = L_i^{fwd}$, and $W_i(t)$ is a Brownian motion in the measure $\mathbb{P}_{i+1}$

*Problem*: each Libor $L_i(t)$ is described in a different measure.

We would like to describe the joint dynamics of all rates in a common measure.
This line of argument leads to the Libor Market Model. Simpler approach: short rate models
Describe the joint distribution of the Libors $L_i(t_i)$ at their setting times $t_i$

$$L_0 \xrightarrow{\phantom{i}} L_1 \xrightarrow{\phantom{i}} L_i \xrightarrow{\phantom{i}} L_{n-2} \xrightarrow{\phantom{i}} L_{n-1}$$

$$0 \quad 1 \quad \ldots \quad i \quad i+1 \quad \ldots \quad n$$

Libors (short rate) $L_i(t_i)$ are log-normally distributed

$$L_i(t_i) = \tilde{L}_i e^{\sigma_i x(t_i) - \frac{1}{2} \sigma_i^2 t_i}$$

where $\tilde{L}_i$ are constants to be determined such that the initial yield curve is correctly reproduced (calibration)

$x(t)$ is a Brownian motion. A given path for $x(t)$ describes a particular realization of the Libors $L_i(t_i)$
Black, Derman, Toy model

The model is formulated in the “spot” measure, where the numeraire $B(t)$ is the discrete version of the money market account.

\[
\begin{align*}
B(t_0) &= 1 \\
B(t_1) &= 1 + L_0 \tau \\
B(t_2) &= (1 + L_0 \tau)(1 + L_1 \tau) \\
&\vdots
\end{align*}
\]

*Model parameters:*

- Volatility of Libor $L_i$ is $\sigma_i$
- Coefficients $\tilde{L}_i$

The volatilities $\sigma_i$ are calibrated to the caplet volatilities (e.g. ATM vols), and the $\tilde{L}_i$ are calibrated such that the yield curve is correctly reproduced.
Price of a zero coupon bond paying $1 at time $t_i$ is given by an expectation value in the spot measure

$$P_{0,i} = \mathbb{E}\left[\frac{1}{B(t_i)}\right] = \mathbb{E}\left[\frac{1}{(1 + L_0 \tau)(1 + L_1 (x_1) \tau) \cdots (1 + L_{i-1} (x_{i-1}) \tau)}\right]$$

The coefficients $\tilde{L}_j$ can be determined by a forward induction:

$$P_{0,1} = \frac{1}{1 + \tilde{L}_0 \tau} \rightarrow \tilde{L}_0$$

$$P_{0,2} = P_{0,1} \mathbb{E}\left[\frac{1}{1 + L_1 (x_1) \tau}\right] = P_{0,1} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t_1}} e^{-\frac{1}{2t_1} x^2} \frac{1}{1 + L_1 e^{\psi_1 x - \frac{1}{2} \psi_1^2 t_1 \tau}}$$

$$\rightarrow \tilde{L}_1$$

and so on... Requires solving a nonlinear equation at each time step.
The model was originally formulated on a tree. Discretize the Markovian driver $x(t)$, such that from each $x(t)$ it can jump only to two values at $t = t + \tau$.

**Mean and variance**

\[
\mathbb{E}[x(t + \tau) - x(t)] = \frac{1}{2} \sqrt{\tau} - \frac{1}{2} \sqrt{\tau} = 0
\]

\[
\mathbb{E}[(x(t + \tau) - x(t))^2] = \frac{1}{2} \tau + \frac{1}{2} \tau = \tau
\]

Choose $x_{up} = x + \sqrt{\tau}$, $x_{down} = x - \sqrt{\tau}$ with equal probabilities such that the mean and variance of a Brownian motion are correctly reproduced.
**BDT tree - Markov driver $x(t)$**

*Inputs:*

1. Zero coupon bonds $P_{0,i}$
   Equivalent with zero rates $R_i$
   
   $P_{0,i} = \frac{1}{(1 + R_i)^i}$

2. Caplet volatilities $\sigma_i$

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Phase Transition in a Log-Normal Interest Rate Model
**BDT tree - the short rate** $r(t)$

**Calibration:** Determine $r_0, r_1, r_2, \cdots$ such that the zero coupon prices are correctly reproduced.

Zero coupon bonds $P_{0,i}$

$$P_{0,i} = \mathbb{E} \left[ \frac{1}{B_i} \right]$$

Money market account $B(t)$

- node and path dependent

$$B_0 = 1$$
$$B_1 = 1 + r(1)$$
$$B_2 = (1 + r(1))(1 + r(2))$$
Calibration in detail

$t = 1$: no calibration needed

\[ P_{0,1} = \frac{1}{1 + r_0} \]

$t = 2$: solve a non-linear equation for \( r_1 \)

\[ P_{0,2} = \frac{1}{1 + r_0} \left( \frac{1}{2} \frac{1}{1 + r_1 e^{\sigma_1 - \frac{1}{2} \sigma_1^2}} + \frac{1}{2} \frac{1}{1 + r_1 e^{-\sigma_1 - \frac{1}{2} \sigma_1^2}} \right) \]

and so on for \( r_2, \ldots \).

Once \( r_i \) are known, the tree can be populated with values for the short rate \( r(t) \) at each time \( t \).

Products (bond options, swaptions, caps/floors) can be priced by working backwards through the tree from the payoff time to the present.
Pricing a zero coupon bond maturing at $t = 2$

We know $r_{1}^{up} = r_{1} e^{\sigma_{1} - \frac{1}{2} \sigma_{1}^2}$ and $r_{1}^{down} = r_{1} e^{-\sigma_{1} - \frac{1}{2} \sigma_{1}^2}$ → can find the bond prices $P_{t,2}$ for all $t$.
BDT model in the terminal measure
Keep the same log-normal distribution of the short rate $L_i$ as in the BDT model, but work in the terminal measure

$$L_i(t_i) = \tilde{L}_i e^{\psi_i x(t_i) - \frac{1}{2} \psi_i^2 t_i}$$

$L_i(t_i) =$ Libor rate set at time $t_i$ for the period $(t_i, t_{i+1})$

Numeraire in the terminal measure: $P_{t,n}$, the zero coupon bond maturing at the last time $t_n$
Why the terminal measure?

Why formulate the Libor distribution in the terminal measure?

- **Numerical convenience.** The calibration of the model is simpler than in the spot measure: no need to solve a nonlinear equation at each time step.

- The model is a particular parametric realization of the so-called Market functional model (MFM), which is a short rate model aiming to reproduce exactly the caplet smile. MFM usually formulated in the terminal measure.

- More general functional distributions can be considered in the Markov functional model \( L_i(t_i) = \tilde{L}_i f(x_i) \), parameterized by an arbitrary function \( f(x) \). This allows more general Libor distributions.
1. The BDT model in the terminal measure can be solved analytically for the case of uniform Libor volatilities $\psi_i = \psi$. Solution possible (in principle) also for arbitrary $\psi_i$, but messy results.

2. The analytical solution has a surprising behaviour at large volatility:
   - The convexity adjustment explodes at a critical volatility, such that the average Libors in the terminal measure (convexity-adjusted Libors) $\tilde{L}_i$ become tiny (below machine precision)
   - This is very unusual, as the convexity adjustments are “supposed” to be well-behaved (increasing) functions of volatility
   - The Libor distribution function in the natural measure collapses to very small values (plus a long tail) above the critical volatility
   - Caplet volatility has a jump at the critical volatility, after which it decreases slightly
What do we expect to find?

Convexity adjusted Libor $\tilde{L}_i$ = related to the price of an instrument paying $L_i(t_i, t_{i+1})$ set at time $t_i$ and paid at time $t_n$

$$\text{Price} = P_{0,n} E_n[L_i] = P_{0,n} \tilde{L}_i$$

Floating payment with delay: the convexity adjustment depends on the correlation between $L_i$ and the delay payment rate $L(t_{i+1}, t_n)$

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$^2$Argument due to Radu Constantinescu.
Consider an instrument paying the rate $L_{ab}$ at time $c$. The price is proportional to the average of $L_{ab}$ in the $c$-forward measure

$$\text{Price} = P_{0,c} E_c [L_{ab}]$$

Can be computed approximatively by assuming log-normally distributed $L_{ab}$ and $L_{bc}$ in the $b$-forward measure, with correlation $\rho$

$$E_c[L_{ab}] \approx L_{ab}^{\text{fwd}} \left( 1 - L_{bc}^{\text{fwd}} (c - b) (e^{\rho \sigma_{ab} \sigma_{bc} \sqrt{ab}} - 1) + O((L_{bc}^{\text{fwd}}(c - b))^2) \right)$$
Convexity adjustment

The convexity adjustment is negative if the correlation $\rho$ between $L_{ab}$ and $L_{bc}$ is positive.

$$\mathbb{E}_c[L_{ab}] \simeq L_{ab}^{\text{fwd}} \left(1 - L_{bc}^{\text{fwd}}(c - b)(e^{\rho \sigma_{ab} \sigma_{bc} \sqrt{ab}} - 1) + O((L_{bc}^{\text{fwd}}(c - b))^2)\right)$$

Convexity-adjusted Libors $\tilde{L}_i$ for 3m Libors ($b = a + 0.25$)

Parameters:

- $\sigma_{ab} = \sigma_{bc} = 40\%$
- $\rho = 20\%$, $c = 10$ years

Naive expectation: The convexity adjustment is largest in the middle of the time simulation interval, and vanishes near the beginning and the end.
Convexity adjusted Libors - analytical solution

The solution for the convexity adjusted Libors $\tilde{L}_i$ for several values of the volatility $\psi$

Simulation parameters: $L_{i}^{\text{fwd}} = 5\%$, $n = 40$, $\tau = 0.25$
Surprising results

- For sufficiently small volatility $\psi$, the convexity-adjusted Libors $\tilde{L}_i$ agree with expectations from the general convexity adjustment formula.
- For volatility larger than a critical value $\psi_{cr}$, the convexity adjustment grows much faster.
- The model has two regimes, of low and large volatility, separated by a sharp transition.
- Practical implication: the convexity-adjusted Libors $\tilde{L}_i$ become very small, below machine precision, and the simulation truncates them to zero.
The size of the convexity adjustment is given by the expectation value

\[ N_i = \mathbb{E}[\hat{P}_{i,i+1} e^{\psi x - \frac{1}{2} \psi^2 t_i}] \]

Recall that the convexity-adjusted Libors are \( \tilde{L}_i = L_i^{\text{fwd}} / N_i \)

Plot of log \( N_i \) vs the volatility \( \psi \)

Simulation with \( n = 40 \) quarterly time steps

\( i = 30, t = 7.5, r_0 = 5\% \)

Note the sharp increase after a critical volatility \( \psi_{cr} \sim 0.33 \)
Motivation
Black, Derman, Toy model
Model with log-normal rates in the terminal measure
Phase Transition

Explanation

The expectation value as integral

\[ N_i = \mathbb{E}[\hat{P}_{i,i+1} e^{\psi x - \frac{1}{2} \psi^2 t_i}] = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t_i}} e^{-\frac{1}{2t_i}x^2} \hat{P}_{i,i+1}(x)e^{\psi x - \frac{1}{2} \psi^2 t_i} \]

The integrand

Simulation with \( n = 20 \) quarterly time steps \( i = 10, t_i = 2.5 \)

\( \psi = \begin{cases} 0.4 & \text{(solid)} \\ 0.5 & \text{(dashed)} \\ 0.52 & \text{(dotted)} \end{cases} \)

Note the secondary maximum which appears for super-critical volatility at \( x \sim 10\sqrt{t_i} \). This will be missed in usual simulations of the model.
More surprises: Libor probability distribution

Above the critical volatility $\psi > \psi_{cr}$, the Libor distribution in the natural measure collapses to very small values, and develops a long tail.

Simulation with $n = 40$ quarterly time steps $\tau = 0.25$. The plot refers to the Libor $L_{30}$ set at time $t_i = 7.5$. 

![Graph showing the Libor probability distribution with different values of $\psi$. The distribution collapses to very small values above the critical volatility $\psi_{cr}$.](image-url)
Caplet Black volatility

Simulation with $n = 40$ quarterly time steps $\tau = 0.25$. The plot refers to the Libor $L_{30}$ set at time $t_i = 7.5$

For small volatilities, the ATM caplet vol is equal with the Libor vol $\psi_i$. Above the critical volatility, the ATM caplet vol increases suddenly

$$\sigma_{LN}^2 t_i = \log \left( \frac{\mathbb{E}[L_i^2]}{\mathbb{E}[L_1]^2} \right)$$
Caplet smile

Above the critical volatility the caplet implied volatility develops a smile

Simulation with $n = 40$ quarterly time steps $\tau = 0.25$. The plot refers to the Libor $L_{30}$ set at time $t_i = 7.5$. 
Conclusions

- For sufficiently small volatility, the model with log-normally distributed Libors in the terminal measure produces a log-normal caplet smile.
- The probability distribution for the Libors in the natural (forward) measure is log-normal.
- Above a critical volatility $\psi_{cr}$ the Libor probability distribution collapses at very small values, and develops a fat tail.
- The caplet Black volatility increases suddenly above the critical volatility, and develops a smile.
- These effects are due to a coherent superposition of convexity adjustments.

In practice we would like to use the model only in the sub-critical regime. Under what conditions does this transition appear, and how can we find the critical volatility?
Phase Transition
The generating function

We would like to investigate the nature of the discontinuous behaviour observed at the critical volatility, and to calculate its value.

Define a generating function for the coefficients $c_j^{(i)}$ giving the one-step zero coupon bond

$$f^{(i)}(x) = \sum_{j=0}^{n-i-1} c_j^{(i)} x^j$$

This was motivated by a simpler solution for the recursion relation.

The expectation value which displays the discontinuity is simply

$$N_i = \sum_{j=0}^{n-i-1} c_j^{(i)} e^{j\psi^2 t_i} = f^{(i)}(e^{\psi^2 t_i})$$
Properties of the generating function

\[ f^{(i)}(x) \text{ is a polynomial in } x \text{ of degree } n - i - 1 \]

\[ f^{(i)}(x) = 1 + c_1^{(i)}x + c_2^{(i)}x^2 + \cdots + c_{n-i-1}^{(i)}x^{n-i-1} \]

where the coefficients are all positive and decrease with \( j \)

Can be found in closed form in the zero and infinite volatility limits \( \psi \to 0, \infty \)

It has no real positive zeros, but has \( n - i - 1 \) complex zeros. They are located on a curve surrounding the origin.
Complex zeros of the generating function

Example: simulation with $n = 40$ quarterly time steps $\tau = 0.25$, flat forward short rate $r_0 = 5\%$

![Diagram of complex zeros]

The zeros of the generating function at $i = 30$, corresponding to the Libor set at $t_i = 7.5$ years

Number of zeros $= n - i - 1 = 9$

Volatility $\psi = 30\%$

*Key mathematical result:* The generating function $f^{(i)}(x)$ is continuous but its derivative has a jump at the point where the zeros pinch the real positive axis.
Complex zeros - volatility dependence

Criterion for determining the critical volatility: The critical point at which the convexity adjustment increases coincides with the volatility where the complex zeros cross the circle of radius $R_1 = e^{\psi^2 t_i}$
Complex zeros - effect on caplet volatility

Simulation with $n = 40$ quarterly time steps, at $i = 30$

The turning point in $\sigma_{ATM}$ coincides with the volatility where the complex zeros cross the circle of radius $R_1 = e^{\psi^2 t_i}$
Phase transition

The model has discontinuous behaviour at a critical volatility $\psi_{cr}$

- The critical volatility at time $t_i$ is given by that value of the model volatility $\psi$ for which the complex zeros of the generating function $f^{(i)}(z)$ cross the circle of radius $e^{\psi^2 t_i}$
- The position of the zeros and thus the critical volatility depend on the shape of the initial yield curve $P_{0,i}$
- For a flat forward short rate $P_{0,i} = e^{-r_0 t_i}$ the zeros are $z_k = e^{-r_0 \tau} x_k$, where $x_k$ are the complex zeros of the simple polynomial

$$P_n(x) = \frac{1}{1 - e^{-r_0 \tau}} + x + x^2 + \cdots + x^{n-i-1}$$

- Approximative solution for the critical volatility at time $t_i$

$$\psi_{cr}^2 \simeq \frac{1}{i(n-i-1)\tau} \log \left( \frac{1}{r_0 \tau} \right)$$
Phase transition - qualitative features

- The critical volatility decreases as the size of the time step $\tau$ is reduced, approaching a very small value in the continuum limit.

- The critical volatility increases as the forward short rate $r_0$ is reduced, approaching a very large value as the rate $r_0$ becomes very small → the applicability range of the model is wider in the small rates regime.

The phenomenon is very similar with a phase transition in condensed matter physics, e.g. steam-liquid water condensation/evaporation, or water freezing.

The Lee-Yang theory of phase transitions relates such phenomena to the properties of the complex zeros of the grand canonical partition function.
Numerical results

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**Table:** The maximal Libor volatility $\psi$ for which the model is everywhere below the critical volatility $\psi_{cr}$, for several choices of the total tenor $t_n$, time step $\tau$ and the level of the interest rates $r_0$. 
Conclusions

- The Black, Derman, Toy model with log-normally distributed Libor in the terminal measure can be solved exactly in the limit of constant and uniform rate volatility.
- The analytical solution shows that the model has two regimes at low- and high-volatility, with very different qualitative properties.
- The solution displays discontinuous behaviour at a certain critical volatility $\psi_{cr}$.
- Low volatility regime
  - Log-normal caplet smile
  - Well-behaved Libor distributions
- High volatility regime
  - The convexity adjustment explodes
  - Libor pdf is concentrated at very small values, and has a long tail
  - A non-trivial caplet smile is generated.
Comments and questions

- A similar behaviour is expected also in a model with non-uniform Libor volatilities (time dependent), but the form of the analytical solution is more complicated.

- Is this phenomenon generic for models with log-normally distributed short rates, e.g. the BDT model, or is it rather a consequence of working in the terminal measure?

- Are there other interest rate models displaying similar behaviour?
BDT model in the terminal measure - calibration and exact solution
Calibrating the model by backward recursion

The non-arbitrage condition in the terminal measure tells us that the zero coupon bonds divided by the numeraire should be martingales

\[
P_{i,j} \over P_{i,n} = \mathbb{E} \left[ \frac{1}{P_{j,n}} | \mathcal{F}_i \right], \text{ for all pairs } (i, j)
\]

Denote the numeraire-rebased zero coupon bonds as

\[
\hat{P}_{i,j} = \frac{P_{i,j}}{P_{i,n}}
\]

Choose the two cases

\[ (i, j) = (i, i + 1) \]

\[
\hat{P}_{i,i+1}(x_i) = \mathbb{E}[\hat{P}_{i+1,i+2}(1 + L_{i+1}\tau) | \mathcal{F}_i]
\]

\[ (i, j) = (0, i) \]

\[
\hat{P}_{0,i} = \mathbb{E}[\hat{P}_{i,i+1}(1 + L_i\tau)]
\]
The two non-arbitrage relations can be used to construct recursively $\hat{P}_{i,i+1}(x_i)$ and $\tilde{L}_i$ working backwards from the initial conditions

$$\hat{P}_{n-1,n}(x) = 1, \quad \tilde{L}_{n-1} = L_{n-1}^{\text{fwd}}$$

No root finding is required at any step.

The calculation of the expectation values requires an integration over $x_{i+1}$ at each step. Usual implementation methods:

1. Tree. Construct a discretization for the Brownian motion $x(t)$
2. SALI tree. Interpolate the function $\hat{P}_{i,i+1}(x_i)$ between nodes and perform the integrations numerically
3. Monte Carlo implementation
Analytical solution for uniform $\psi$

Consider the limit of uniform Libor volatilities $\psi_i = \psi$
The model can be solved in closed form starting with the ansatz

$$\hat{P}_{i,i+1}(x) = \sum_{j=0}^{n-i-1} c_j^{(i)} e^{j \psi x - \frac{1}{2} j^2 \psi^2 t_i}$$

Matrix of coefficients $c_j^{(i)}$ is triangular (e.g. for $n = 5$)

$$\hat{c} = \begin{pmatrix}
  c_0^{(n-1)} & 0 & 0 & 0 & 0 \\
  c_0^{(n-2)} & c_1^{(n-2)} & 0 & 0 & 0 \\
  c_0^{(n-3)} & c_1^{(n-3)} & c_2^{(n-3)} & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  c_0^{(1)} & c_1^{(1)} & c_2^{(1)} & c_3^{(1)} & 0 \\
  c_0^{(0)} & c_1^{(0)} & c_2^{(0)} & c_3^{(0)} & c_4^{(0)}
\end{pmatrix}$$
Recursion relation for the coefficients $c_j^{(i)}$

The coefficients $c_j^{(i)}$ and convexity adjusted Libors $\tilde{L}_i$ can be determined by a recursion

$$c_j^{(i)} = c_j^{(i+1)} + \tilde{L}_{i+1}\tau c_j^{(i+1)} e^{(j-1)\psi^2 t_{i+1}}$$

$$\tilde{L}_i = \frac{\hat{P}_{0,i} - \hat{P}_{0,i+1}}{\tau \sum_{j=0}^{n-i-1} c_j^{(i)} e^{j\psi^2 t_i}} = L_i^{\text{fwd}} \frac{\hat{P}_{0,i+1}}{\sum_{j=0}^{n-i-1} c_j^{(i)} e^{j\psi^2 t_i}}$$

starting with the initial condition

$$\tilde{L}_{n-1}\tau = \hat{P}_{0,n-1} - 1, \quad \hat{P}_{n-1,n}(x) = 1$$
Recursion for the coefficients $c_j^{(i)}$

Linear recursion for the coefficients $c_j^{(i)}$. They can be determined backwards from the last time point starting with $c_0^{(n-1)} = 1$

$$c_j^{(i)} = c_j^{(i+1)} + \tilde{L}_{i+1} \tau c_{j-1}^{(i+1)} e^{(j-1)\psi^2 t_{i+1}}$$

$i = n - 1$:

$$c_0^{(n-1)}$$

$i = n - 2$:

$$c_0^{(n-2)} \quad c_1^{(n-2)}$$

$i = n - 3$:

$$c_0^{(n-3)} \quad c_1^{(n-3)} \quad c_2^{(n-3)}$$
Solution of the model

Knowing $\hat{P}_{i,i+1}(x)$ and $\tilde{L}_i$ one can find all the zero coupon bond prices

$$P_{i,j}(x) = \frac{\hat{P}_{i,j}(x)}{\hat{P}_{i,i+1}(x)(1 + \tilde{L}_i \tau_i e^{\psi x - \frac{1}{2} \psi^2 t_i})}$$

where

$$\hat{P}_{i,j}(x) = \mathbb{E}\left[ \frac{1}{P_{j,n}} | \mathcal{F}_i \right] = \mathbb{E}[\hat{P}_{j,j+1}(1 + \tilde{L}_j \tau_j e^{\psi x_j - \frac{1}{2} \psi^2 t_j}) | \mathcal{F}_i]$$

$$= \sum_{k=0}^{n-j-1} c^{(j)}_k e^{k \psi x - \frac{1}{2} (k \psi)^2 t_i}$$

$$+ \tilde{L}_j \tau_j \sum_{k=0}^{n-j-1} c^{(j)}_k e^{(k+1) \psi x - \frac{1}{2} (k^2+1) \psi^2 t_i + k \psi^2 (t_j - t_i)}$$

All Libor and swap rates can be computed along any path $x(t)$.