1. Automated Option Market Making

2. Local Variance Gamma Model

3. Conclusions
As markets move from an open outcry system to computerized exchanges, the question arises as to how to automate the pricing and risk management of derivative securities.

We take the view that this problem is presently unsolved and this explains why firms continue to use traders to price and hedge.

The true “revenge of the nerds” will begin when a quant develops a solution to this problem.
Banks, software vendors, and other options market participants have historically hired armies of PhD’s to tackle the following apparently simple problem.

**Problem Statement:** Given the current market price of the underlying asset and also given market option quotes at several given strikes and terms, provide options quotes at any strike and term in a specified set.

For simplicity, we suppose that both the supplied quotes and the produced quotes are mid-points of the bid and the ask.

For simplicity, we suppose that both the input quotes and the output quotes are for European options.

We will refer to the given strikes, terms, and mids as “listed”. The problem of inferring option prices at non-listed strikes and terms arises on both exchanges and OTC.
Since the supplied mids can be depicted as points in the first octant of $\mathbb{R}^3$, it first appears that armies of PhD’s are merely being asked to connect the dots.

As my 7 year old daughter mastered this intellectual challenge several years ago, one has to wonder if this is not the financial equivalent of the lightbulb joke.

It is my contention that once all the desired properties of the solution are clarified, no one has ever published a solution that successfully connects the dots.
Recall that the option market-makers’ pricing problem only asks that option prices be provided for strikes and terms in a specified set.

When this set lies in the convex hull of listed strikes and terms, the option market makers pricing problem can be properly described as “glorified interpolation”.

When at least part of the desired set lies beyond listed strikes and terms, extrapolation is involved. Philosophically, there is no unique solution to any extrapolation problem.

What makes this problem even more difficult is that there is a long laundry list of desirable properties that a solution is asked to possess. Unfortunately, these properties are typically only discovered after much effort has been spent solving the wrong problem.
A Partial List of Desirable Properties

The following highly demanding criteria can usually be expressed in any of several mathematically equivalent spaces. Common spaces used for the output include option price space, implied vol space, and local vol space. For simplicity, we treat the independent variables as strike $K$ and term $T$.

- robust to input data, i.e. desirable output is produced no matter what is input.
- well-posedness, i.e. small changes in inputs lead only to small changes in output.
- locality, i.e. a small change in one input has only a local effect on the output.
- ability to accept related inputs eg. American options, variance swaps, CDS, options on other underlyings.
- no model-free arbitrages in the output (assuming wide support for possible levels of the underlying uncertainty).
- exact price consistency with given mids (implied by above).
- smooth output (except for option prices at $T = 0$).
- perfect out-of-sample prediction in $K, T$ space.
- ability to extrapolate in $S, t$ space i.e., can produce deltas, gammas, thetas, spot slides, and time slides.
- resonance with financial orthodoxy eg. nonnegative gammas.
• perfect out-of-sample prediction in the Cartesian product of spot prices $S$ and calendar time $t$.

• internal consistency eg. quantities assumed constant in a parametric model should not vary with $S$, $t$ or other observables.

• real-time computational speed.

• numerical robustness (always works on finite precision computers).
ability to uniquely and accurately price related derivatives such as American options, variance swaps, barrier options, and other exotics.

parsimony

distinct economic role for parameters that permit identification.

implied $\mathbb{Q}$ dynamics not ridiculously far from observable $\mathbb{P}$ dynamics - no stat. arb.
Local Variance Gamma (henceforth LVG) is a work in progress which satisfies many, but not all, of the criteria just mentioned. The model takes its name from the fact that it simultaneously generalizes both Dupire’s 1994 Local Variance model and Madan and Seneta’s 1990 Variance Gamma model.

The use of the word “local” in the LVG name is apt for two reasons:

1. As in local vol models, the uncertainty in stock returns is assumed to depend (only) on the stock price level and time.
2. Calibration is local, if \( N \) option prices are given, one does \( N \) univariate searches, rather than one global search for \( N \) parameters.
The input into LVG is any set of arbitrage-free option prices at discrete strikes and maturities on a regular grid. The output is an exactly-consistent arbitrage-free price process for the underlying asset.

Others have also done this; what is special about LVG AFAIK is that calibration to discrete strike and maturity option prices is local and yet the produced dynamics for the underlying asset price evolve in cont. time with a cont. state space.
In LVG, the underlying forward price process is assumed to be a driftless diffusion running on an independent gamma clock. The resulting continuous-time price process is a pure jump Markov martingale with infinite activity.

The LVG price process is not a Lévy process, but it enjoys time & space homogeneity between listed strikes & terms.

If option prices at just 1 term are given, the process is time-homogeneous.

Likewise, if option prices at just 1 strike per term are given, the process is space-homogeneous (given by time-dependent VG).

When 2 or more option prices at 1 term or strike are given, the diffusion coefficient is assumed to be piecewise constant, i.e., it jumps at listed strikes and maturities, but is otherwise flat.
Assumptions

- In this presentation, I will just show the case where option prices at a single term are given.
- Furthermore, interest rates and dividends are zero and limited liability is not respected. (all of this can be relaxed).
Let \( D \) be a time homogeneous diffusion starting at \( S_0 \):

\[
dD_s = a(D_s) \, dW_s, \quad s \geq 0,
\]

where the diffusion coefficient \( a(x) : \mathbb{R} \mapsto \mathbb{R} \) is initially unspecified.

Let \( \{\Gamma_t, t > 0\} \) be an independent gamma process with Lévy density

\[
k_{\Gamma}(t) = \frac{e^{-\alpha t}}{t^{*} t}, \quad t > 0 \text{ for parameters } t^{*} > 0 \text{ and } \alpha > 0.
\]

The marginal distribution of a gamma process at time \( t \geq 0 \) is a gamma distribution with PDF:

\[
Q\{\Gamma_t \in ds\} = \frac{\alpha^{t/t^*}}{\Gamma(t/t^*)} s^{t/t^* - 1} e^{-\alpha s}, \quad s > 0, \alpha > 0, t^* > 0.
\]
Recall that the marginal distribution of a gamma process at time \( t \geq 0 \) is a gamma distribution with PDF:

\[
\mathbb{Q}\{ \Gamma_t \in ds \} = \frac{\alpha^{t/t^*}}{\Gamma(t/t^*)} s^{t/t^* - 1} e^{-s} \cdot s > 0, \alpha > 0, t^* > 0.
\]

We set the parameter \( \alpha = \frac{1}{t^*} \), so that the gamma process becomes unbiased, i.e. \( \mathbb{E}^\mathbb{Q}\Gamma_t = t \) for all \( t \geq 0 \).

The PDF of the unbiased gamma process \( \Gamma \) at time \( t \geq 0 \) is:

\[
\mathbb{Q}\{ \Gamma_t \in ds \} = \frac{s^{t/t^* - 1} e^{-s}}{(t^*)^{t/t^*} \Gamma(t/t^*)}, \quad s > 0,
\]

for parameter \( t^* > 0 \).

When \( t = t^* \), this PDF is exponential. We choose \( t^* \) to be the given term, so that the distribution of the gamma clock at expiry is exponential.
We assume that the underlying spot price process $S$ is obtained by subordinating the driftless diffusion $D$ to the unbiased gamma process $\Gamma$:

$$S_t = D_{\Gamma_t}, \quad t \geq 0.$$

- $S$ inherits the local martingale property from the diffusion $D$.
- $S$ inherits dependence of its increments on its level from $D$.
- $S$ inherits jumps from the gamma process $\Gamma$.
- $S$ inherits time homogeneity from $D$ and $\Gamma$. 
Define \( C(S, T, K) \) as the value of a European call with stock price \( S \in \mathbb{R} \), term \( T \geq 0 \), and strike \( K \in \mathbb{R} \) in the LVG model.

Then it can be shown that the LVG model value satisfies the following forward PDDE:

\[
\frac{C(S, T + t^*, K) - C(S, T, K)}{t^*} = \frac{a^2(K)}{2} \frac{\partial^2}{\partial K^2} C(S, T + t^*, K),
\]

for all \( S \in \mathbb{R}, T \geq 0, \) and \( K \in \mathbb{R} \).

Remarkably, the effect of the gamma time change is to simultaneously inject realism into the underlying price dynamics and to semi-discretize Dupire’s forward PDE.

More specifically, Dupire’s infinitessimal calendar spread gets replaced by a discrete calendar spread, with the time between the two maturities given by \( t^* \).
Setting the first maturity date to the valuation date and $S = S_0$, the LVG model value of a call satisfies:

$$\frac{C(S_0, t^*, K) - (S_0 - K)^+}{t^*} = \frac{a^2(K)}{2} \frac{\partial^2}{\partial K^2} C(S_0, t^*, K),$$

for all $K \in \mathbb{R}$.

If call prices at a continuum of strikes are given, then the diffusion coefficient $a(x)$ becomes observed.

As a result, the stock price process becomes specified. Options of other terms and other derivatives can be valued by Monte Carlo.
Consider a European-style claim paying $\phi(S_T)$ at its maturity date $T \geq t^*$. Let $V(S, t) : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ denote the function that relates the value of this claim to the underlying spot price $S$ and calendar time $t$. It can be shown that $V(S, t)$ solves the following backward PDDE:

$$
\frac{a^2(S)}{2} \frac{\partial^2}{\partial S^2} V(S, t) = \frac{V(S, t) - V(S, t + t^*)}{t^*}.
$$

We note that this PDDE arises if Rothe’s method (also known as semi-discretization or (horizontal) method of lines) is used to discretize the diffusion equation.
The European claim value function $V$ satisfies the terminal condition:

$$V(S, T) = \phi(S), \quad S \in \mathbb{R}. $$

Re-arranging the backward PDDE implies that $V(S, t)$ solves the following second order inhomogeneous ODE:

$$t^* \frac{a^2(S)}{2} \frac{\partial^2}{\partial S^2} V(S, t) - V(S, t) = -V(S, t + t^*).$$

Evaluating at $t = T - t^*$ the RHS forcing term is $\phi(S)$. One can numerically solve the ODE and recursively step backwards in time steps of length $t^*$.

The algorithm switches from the express (ODE) to the local (PIDE), when the express would overshoot the valuation time.
Setting $T = 0$ and $S = S_0$ in the forward PDDE, the LVG model value of a call solves a linear second order ODE in $K$:

$$t^* \frac{a^2(K)}{2} \frac{\partial^2}{\partial K^2} C(S_0, t^*, K) - C(S_0, t^*, K) = -(S_0 - K)^+$$

for all $K \in \mathbb{R}$.

When $a(K) = a$ is a constant, one can solve the call option value analytically,

$$C(K) = (S_0 - K)^+ + \frac{1}{2\beta} e^{-\beta |S_0 - K|}, \quad \beta = \sqrt{\frac{2}{a^2 t^*}}. \quad (1)$$

The time value of the option is a symmetric, double-exponential function of strike, centered at $K = S_0$.

ATM option is proportional to square root of maturity.
Under constant percentage volatility, \( a(K) = \sigma K \), we can define \( k \equiv \ln K \), \( s_0 \equiv \ln S_0 \) and apply change of variables to obtain,

\[
\frac{1}{2} \sigma^2 t^* (C_{kk} - C_k) - C = - \left( e^{s_0} - e^k \right)^+. \tag{2}
\]

The solution is,

\[
C(k) = (S_0 - K)^+ + \frac{S_0}{(\beta_+ + \beta_-)} e^{-\beta_+ |s_0 - k|}, \tag{3}
\]

where

\[
\beta_\pm = \sqrt{\frac{1}{4} + \frac{2}{\sigma^2 t^*}} \pm \frac{1}{2}.
\]

\( \beta_+ \) is for \( k > s_0 \) and \( \beta_- \) is for \( k \leq s_0 \).

The time value of the option is an asymmetric, double-exponential function of log strike, centered at \( K = S_0 \).

The asymmetry increases with \( \sigma \) and maturity.
**Implied Volatility Smiles/Skews from LVG model**

**Constant absolute volatility:**  \( a(K) = a = 20\% \)

Maturity: 1 months  
Maturity: 12 months  
Maturity: 24 months

**Constant percentage volatility:**  \( a(k) = \sigma K, \sigma = 20\% \)
PDF of Log Return Under Constant $\sigma$ LVG model

- Maturity: 1 months
- Maturity: 12 months
- Maturity: 24 months

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Local Variance Gamma
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LVG Volatility Calibration

- From the forward PDDE (percentage vol representation), we have,

\[ \sigma^2(k, T + t^*) = \frac{2}{\tau} \frac{C(K, T + t^*) - C(K, T)}{K^2 C_{KK}(K, T + t^*)}. \] (4)

- To calibrate Dupire’s LV model, one needs both continuous strike (for \( C_{kk}, C_k \)) and continuous maturity (for \( C_T \)).

- To calibrate our LVG model, we only need continuous strike.

  - Observations along strikes are much denser than along maturities.
  - Interpolation along strikes is much easier.
  - One way to achieve stable interpolation is to interpolate the implied volatility smile directly:

\[ \sigma^2(k) = \frac{2\Sigma}{t^*} \frac{C(K, T + t^*) - C(K, T)}{SN(d_1) \left( K^2 \Sigma (\Sigma_{KK} - d_1 \Sigma^2_K) + (1 + d_1 K \Sigma_K)^2 \right)}, \]

with \( \Sigma = IV \sqrt{T + t^*} \).
An Example

- Generate option prices from a jump-diffusion stochastic volatility model. Assume observations at 4 maturities: 1, 6, 12, 24 months; 5 strikes at each maturity: ±20%, ±12%, ±4%.
- Local quadratic fitting on the implied volatility.
- \( t^* \) is set to 1, 5, 6, 12 months to match observation.
- LVG with constant \( \sigma \) generates a smile/skew. To the extent the observed smile/skew is larger/smaller, the LVG volatility can smile or frown.
The LVG model, Entropy Methods, and Mixtures of Lognormals also produce continuous-time arbitrage-free dynamics for the underlying asset price that are exactly consistent with any given set of option prices at discrete strikes and terms.

In terms of smoothness in $K, T, S, t$, LVG lies between Entropy Methods and Mixtures of Lognormals.

Calibration of the LVG process just involves univariate root-finding on one option at a time. In contrast, Entropy Methods and Mixtures of Lognormals both require global searches, which can take a long time or may never succeed.
Parting Remarks

- Just as the gamma function is a natural way to interpolate (and extrapolate) the factorials, subordination to a gamma process is a natural way to interpolate option prices at discrete terms.
- There is no doubt in my mind that LVG can be improved upon in several dimensions, eg. realism, at the possible cost of sacrificing in other dimensions, eg. simplicity.
- In my view, the automated options market making problem remains open.