On notion of arbitrage and robust pricing and hedging of variance swaps

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based on joint works with
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Typically, when modelling, one proceeds as follows:

- Write down a plausible and well behaved model.
- Compute prices of (liquid) financial instruments as function of model parameters.
- Calibrate the model: chose the parameters to match the prices already observed in the market.
- Use it: sell and hedge new derivatives.

This approach has important drawbacks:

- It is exposed to model risk which may be hard to quantify.
- Models are *re-calibrated* daily: theoretically inconsistent.
- Inevitably, it ignores some information present in the market.

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We want to develop a more *robust* approach.
The general challenge for robust approach is as follows:

- **Q1: robust pricing**
  Start with market information: prices of some instruments. Assume it admits no arbitrage ⇐⇒ could come from a model. Given a new product, determine its feasible price, i.e. a price which does not introduce any arbitrage in this market.

- **Q2: robust hedging**
  Furthermore, derive best super-/sub- hedging strategies which always work.

Thus we want to use the information in the market to make statements which are model-independent.

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1 Motivating questions and FTAP with market input
   - Classical vs robust modelling framework
   - General setup and different notions of arbitrage
   - Towards FTAP with market input

2 Weighted variance swaps
   - Standing assumptions
   - $w$-variance swaps and convex payoffs

3 Robust pricing and hedging of options with convex payoffs
   - Main results
   - Upper bound
   - Lower bound
We assume \((S_t : t \leq T)\) takes values in some functional space \(\mathcal{P}\). \(\mathcal{X}\) is a given set of traded assets, mappings from \(\mathcal{P}\) to \(\mathbb{R}\). On this set we have a **pricing operator** \(\mathcal{P}\) which acts linearly on \(\mathcal{X}\), \(\mathcal{P} : \text{Lin}(\mathcal{X}) \rightarrow \mathbb{R}\). \(\mathcal{P}X\) is the market price of \(X\).

Assume interest rates are deterministic, *here set to zero*: \(\mathcal{P}1 = 1\).

We say that there exists a \((\mathcal{P}, \mathcal{X})\)–market model if there is a model \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{Q}, (S_t))\), \(S_t\) a \(\mathcal{Q}\)–martingale and \(\mathcal{P}X = \mathbb{E}_\mathcal{Q}[X], X \in \mathcal{X}\). We would like to have

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\mathcal{P} \text{ admits no arbitrage on } \mathcal{X} \iff \text{there exists a market model} \iff \{\mathcal{P}X\}_{X \in \mathcal{X}} \text{ satisfy some constraints}
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Typically we want to start with simpler \(\mathcal{X}\) and then consider \(\mathcal{X} \cup \{O_T\}\) for an exotic option \(O_T : \mathcal{P} \rightarrow \mathbb{R}\).
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We say that there exists a \((\mathcal{P}, \mathcal{X})\)–market model if there is a model 

\((\Omega, \mathcal{F}, (\mathcal{F}_t), Q, (S_t)), S_t\) a \(Q\)–martingale and \(\mathcal{P}X = E^Q[X]\), \(X \in \mathcal{X}\). We would like to have

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Definition (Model–independent arbitrage)

We say that $\mathcal{P}$ admits a model–independent arbitrage on $\mathcal{X}$ if there exists $X \in \text{Lin}(\mathcal{X})$ with $X \geq 0$ and $\mathcal{P}X < 0$. 
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This coarsest notion is typically sufficient to derive no–arbitrage bounds but not sufficient to give existence of a market model.
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Definition (Weak arbitrage (Davis & Hobson 2007))

We say that $\mathcal{P}$ admits a weak arbitrage on $\mathcal{X}$ if for any model, there exists $X \in \text{Lin}(\mathcal{X})$ with $\mathcal{P}X \leq 0$ but $\mathbb{Q}(X \geq 0) = 1$, $\mathbb{Q}(X > 0) > 0$. 
Theorem (Davis and Hobson (2007))

Let $\mathcal{X} = \{1, (K_i - S_T)^+ : i = 1, \ldots n\}$. The following are equivalent

- $P$ admits no WA on $\mathcal{X}$
- there exists a $(P, \mathcal{X})$-market model
- $p_i = P(K_i - S_t)^+ \geq (K_i - S_0)^+$ and the piecewise linear interpolation of the points $(0, 0), (K_1, p_1), \ldots, (K_{n'}, p_{n'})$ is increasing, convex and with slope strictly bounded by 1, where $n' = \inf\{i : p_i = (K_i - S_0)\} \wedge n$.

where $S_0 = 1$. Note that here $Q(S_T \geq k_{n'}) = 0$ in any market model.
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Definition (Weak free lunch with vanishing risk (Cox & O. 2009))

We say that $\mathcal{P}$ admits a weak free lunch with vanishing risk on $\mathcal{X}$ if there exists $X_n, Z \in \text{Lin}(\mathcal{X})$ such that $X_n \to X$ (pointwise on $\mathcal{P}$), $X_n \geq Z$, $X \geq 0$ and $\lim \mathcal{P}X_n < 0$. 
Theorem (Cox and O. (2009))

Let $\mathcal{X} = \{1, (K - S_T)^+ : K \geq 0\}$. Then the following are equivalent:

1. $\mathcal{P}$ admits no WFLVR on $\mathcal{X}$
2. there exists a $(\mathcal{P}, \mathcal{X})$-market model
3. $P(K) = \mathcal{P}(K - S_T)^+$ satisfies
   
   
   $P(K) \geq (K - S_0)^+$ is convex and non-decreasing, and $P(0) = 0$, $P'(K) \leq 1,$

   $P(K) - (K - S_0) \to 0$ as $K \to \infty,$

   \hspace{1cm} (1)

   
   
   When (1) holds but (2) fails $\mathcal{P}$ admits no model-free arbitrage but a market model does not exist.

Similar thm for $\mathcal{X}$ with digital double barrier options.
Towards general FTAP...

- $\mathcal{H}$ is some functional space and $\mathcal{P}$ is an element of its dual.
- Appropriate no-arbitrage condition is the one which ensures $\mathcal{P}$ extends to a countably additive measure on $\mathcal{B}$.
- Boundary cases (weak arbitrages) correspond to $\mathcal{P}$ being a bounded (finitely) additive measure.
- First step in Cassese (2008): FTAP (for bounded assets) with no probability measure, but with no market input.
- Work in progress...
Our standing assumptions are:

- Liquid market in underlying asset $S_t$, $t \in [0, T]$.
- No transaction costs.
- $(S_t)$ has continuous paths.
- No interest rate volatility.
- Uniquely determined forward price $F_T$ (e.g. deterministic dividend yield).
- Options are traded at time 0 at quoted prices. In this talk all options are European with the same exercise time $T$.

For this talk $r = q = 0$ so that $F_T = S_0$. 
We want to develop robust pricing and hedging for weighed variance swaps. A $w$-weighted variance swap pays

$$V_w^T = \int_0^T w(S_u)d\langle\log S\rangle_u - k^w,$$

where swap rate $k^w$ is set so that $PV_w^T = 0$. We take $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $w(s)/s^2$ locally integrable. Then, in any model,

$$\int_0^T w(S_u)d\langle\log S\rangle_u = 2\lambda_w(S_T) - 2\lambda_w(S_0) - 2\int_0^T \lambda'_w(S_u)dS_u \ a.s ,$$

where $\lambda''_w(s) = w(s)/s^2$.

We have three important examples:

1. **Realised variance swap**: $w \equiv 1$ and $\lambda_w(s) = -\log(s)$.
2. **Corridor variance swap**: $w(s) = 1_{(0,a)}(s)$ or $w(s) = 1_{(a,\infty)}(x)$, where $0 < a < \infty$ and $\lambda_w(s) = (-\log(\frac{s}{a}) + \frac{s}{a} - 1) w(s)$.
3. **Gamma swap**: $w(s) = s$ and $\lambda_w(s) = s \log(s) - s$. 
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Motivating questions and FTAP with market input
Weighted variance swaps
Robust pricing and hedging of options with convex payoffs

Standing assumptions
w-variance swaps and convex payoffs

Lemma

Consider a model \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}, (S_t))\) with \(S_t\) a \(\mathbb{Q}\)-martingale. Then

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\mathbb{E}^\mathbb{Q} \left[ \int_0^T w(S_u) d\langle \log S \rangle_u \right] = 2 \mathbb{E}^\mathbb{Q}[\lambda_w(S_T)] - 2\lambda_w(S_0)
\]

and if they are finite then \(\int_0^T \lambda'_w(S_u) dS_u\) is a value of an admissible self-financing strategy.

Corollary

Suppose vanilla assets \(\mathcal{X}\) with prices \(\mathcal{P}\) are given and there exists a \((\mathcal{P}, \mathcal{X})\)-market model. Then the following are equivalent

- There exists a \((\mathcal{P}, \mathcal{X} \cup \{V^w_T\})\)-market model with \(\mathcal{P}V^w_T = 0\).
- There exists a \((\mathcal{P}, \mathcal{X} \cup \{\lambda_w(S_T)\})\)-market model with \(\mathcal{P}\lambda_w(S_T) = k^w/2 + 2\lambda_w(S_0)\).

Hence we reduce the problem to that of robust pricing and hedging of convex payoffs.
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Hence we reduce the problem to that of robust pricing and hedging of convex payoffs.
Suppose $\mathcal{X} = \{1, (K_i - S_T)^+ : i = 1, \ldots, n\}$ with $\mathcal{P}(K_i - S_t)^+ = p_i$. We are interested in the range of $\mathbb{E}^Q \lambda(S_T)$ over all $(\mathcal{P}, \mathcal{X})$–market models.

*Put differently: we are given prices of $n$ put options and we want to understand no-arbitrage prices (and robust hedges) for an European option with payoff $\lambda(S_T)$.*

The prices only depend on $\mu$ – the risk-neutral law of $S_T$. Given any $\mu$ on $\mathbb{R}_+$ such that

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\int s\mu(ds) = S_0, \quad \int (K_i - s)^+ \mu(ds) = p_i, \ i = 1, \ldots, n
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a $(\mathcal{P}, \mathcal{X})$–market model is given by $S_t = B_{\frac{t}{T-t} \wedge \tau}$, where $B_u$ is a $Q$-BM and $\tau$ solves the Skorokhod embedding problem, $B_T \sim \mu$.

This is also a $(\mathcal{P}, \mathcal{X} \cup \{\lambda(S_T)\})$–market model where $\mathcal{P}\lambda(S_T) = \int \lambda(s)\mu(ds)$.

In particular, no-arbitrage prices of $\lambda(S_T)$ form an interval (by considering random mixtures of models).
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In particular, no-arbitrage prices of $\lambda(S_T)$ form an interval (by considering random mixtures of models).
Suppose \((\mathcal{P}, \mathcal{X})\) do not admit weak arbitrage, \(\lambda'' \geq 0\).

**Primal Problem:** Find

\[
UB_{\lambda} = \sup_{\mu \sim S_T} \int \lambda(s) \mu(ds), \quad LB_{\lambda} = \inf_{\mu \sim S_T} \int \lambda(s) \mu(ds),
\]

over all \((\mathcal{P}, \mathcal{X})\)-market models.

**Dual Problem:** Find

\[
\tilde{UB}_{\lambda} = \inf \left\{ \mathcal{P} \bar{F}(S_T) : \bar{F}(s) = \sum_{i=1}^{n} \pi_i (K_i - s)^+ + \phi s + \psi \geq \lambda(s) \right\}
\]

\[
\tilde{LB}_{\lambda} = \sup \left\{ \mathcal{P} \underline{F}(S_T) : \underline{F}(s) = \sum_{i=1}^{n} \pi_i (K_i - s)^+ + \phi s + \psi \leq \lambda(s) \right\}
\]

**Theorem**

If \( |LB_{\lambda}| < \infty \) then there is no duality gap, \( LB_{\lambda} = \tilde{LB}_{\lambda} \), and there exists an optimal \( F^* \) which solves the dual.

Likewise for the upper bound if there exists at least one superreplicating portfolio.
Suppose \((\mathcal{P}, \mathcal{X})\) do not admit weak arbitrage, \(\lambda'' \geq 0\).

**Primal Problem:** Find range of no-arbitrage prices

**Dual Problem:** Find robust super- and sub- hedges

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Likewise for the upper bound if there exists at least one superreplicating portfolio.

Let \(\mathcal{X}_\lambda = \mathcal{X} \cup \{\lambda(S_T)\}\).

- If \(P_\lambda(S_T) \in (LB_\lambda, UB_\lambda)\) then there exists a \((\mathcal{P}, \mathcal{X}_\lambda)\)–m.m.
- If \(P_\lambda(S_T) \not\in [LB_\lambda, UB_\lambda]\) then there is model-independent arbitrage.
- If \(P_\lambda(S_T) \in \{LB_\lambda, UB_\lambda\}\) then there either exists a \((\mathcal{P}, \mathcal{X}_\lambda)\)–m.m. or there is a weak arbitrage.

\(\tilde{UB}\) and Superreplication — explicit

\(\tilde{LB}\) and Subreplication — solution of a dynamic programming alg
Suppose for simplicity that $\lambda(0) < \infty$ and $\lambda(s) = 0$ for all $s \geq \bar{s}$. Then, in any market model

$$\mathbb{E}^Q[\lambda(S_T)] = \int_0^\infty \lambda''(K)P(K)dK,$$

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Jan Obłój
Lower bound is trickier – there is no uniform lower bound on put prices \( P(K) \) given our market input.

Indeed, choosing minimal price in one interval \([K_i, K_{i+1}]\) typically forces maximal prices in adjacent intervals. It is not clear \textit{a priori} if lower bound is attained and by what measure/put prices, and how to construct a subreplicating portfolio?

- We first showed that it is sufficient to look only at measures with at most \( n + 1 \) atoms,
- then obtained the lower bound as solution to a dynamic programming,
- and then proved it is always a value of a portfolio in market quoted options.

...and finally understood that it all hinges on duality in the theory of semi-infinite programming! (Issi, Karlin 1960)
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For any choice of \(b\), we can form a portfolio of cash, long underlying and short put option with payoff:

\[
x_0, x_1 \text{ solve } \quad g(x_0) = g(x_1) = 1 + b + Ke^{-1-b},
\]

where \(g(x) = \log x + K/x\).

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\bar{F}(S_T) = b + \frac{1}{x_1} S_T - \left(\frac{1}{x_0} - \frac{1}{x_1}\right)(K - S_T)^+ \geq \log S_T,
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and hence \(\mathcal{P}\bar{F}(S_T) = b + \frac{1}{x_1} S_0 - \left(\frac{1}{x_0} - \frac{1}{x_1}\right) p_K \geq \mathcal{P} \log S_T\). Minimising in \(b\) gives the lowest price and the associated superreplication. We then have \(p_K = (K - x_0)(x_1 - S_0)/(x_1 - x_0)\) and hence the bound is attained in a model with \(S_T \sim q\delta_{x_0} + (1 - q)\delta_{x_1}\), with \(q = (x_1 - S_0)/(x_1 - x_0)\).
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In fact the above is nothing else but an semi-infinite linear program:

Find \( v_P = \inf \{ c'z | z \in Z \} \) where

\[
Z = \{ z \in \mathbb{R}^3 : a(s)z \geq b(s) \quad \forall s \in \mathbb{R}^+ \}.
\]

Here \( a(s) \) is the vector of exercise values \( a(s) = (1, s, (K - s)^+) \), \( b(s) = \log s \) and \( c \) is the vector of asset prices \( c = (1, S_0, p_K) \).

Formally the LP dual is:

Find \( v_D = \sup \int_{\mathbb{R}^+} b(s) \mu(ds) \), where the supremum is taken over positive measures \( \mu \) satisfying the equality constraints \( c = \int a(s) \mu(ds) \), i.e.

\[
(1, S_0, p_K) = \left( \int 1 \, d\mu, \int s \, d\mu, \int (K - s)^+ \, d\mu \right).
\]

Our simple calculation shows

- There is no duality gap.
- The fact that the dual maximum is achieved at an atomic measure corresponds to the conventional LP result that dual variables are zero wherever constraints are not binding.
**Numerical example for \( n = 3 \):**

Consider a market input of three European put options maturing in 1 year. The data are \( S_0 = 100 \), \( F_T = 105 \), \( D_T = \exp(-0.03) \), \( K_i = 50, 100 \) and \( 150 \), \( p_1 = 1.127 \), \( p_2 = 18.006 \) and \( p_3 = 53.326 \). The range of (weak) arbitrage-free prices for a vanilla variance swap, corridor variance swap and gamma swap is then:

<table>
<thead>
<tr>
<th>VS type</th>
<th>( w(x) )</th>
<th>( \lambda_w(x) )</th>
<th>Arbitrage bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>VS</td>
<td>1</td>
<td>(-\ln(x))</td>
<td>([0.224, \infty))</td>
</tr>
<tr>
<td>Corr VS</td>
<td>( \mathbf{1}_{\left[\frac{F_T}{75}, \infty\right)}(x) )</td>
<td>((-\ln\left(\frac{xF_T}{75}\right) + \frac{F_Tx}{75} - 1)wh(x))</td>
<td>((0.038, 0.340))</td>
</tr>
<tr>
<td>Gamma S</td>
<td>( x )</td>
<td>(x\ln(x) - x)</td>
<td>((0.125, \infty))</td>
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**Numerical example for** $n = 3$ (cont):

Consider a market input of three European put options maturing in 1 year. The data are $S_0 = 100$, $F_T = 105$, $D_T = \exp(-0.03)$, $K_i = 50, 100$ and $150$, $p_1 = 1.127$, $p_2 = 18.006$ and $p_3 = 53.326$.

The log contract payoff $-\ln(S_T/F_T)$ (blue line) and the consequent sub-hedging portfolio (black line). The portfolio is given by $\pi_1^\dagger = 0.01706$, $\pi_2^\dagger = 0.00472$, $\pi_3^\dagger = 0.00259$, $\phi^\dagger = -0.00536$ and $\psi^\dagger = 0.42517$. 
**Numerical example for** \( n = 3 \) (cont):

Consider a market input of three European put options maturing in 1 year. The data are \( S_0 = 100, \ F_T = 105, \ D_T = \exp(-0.03), \ K_i = 50, 100 \) and 150, \( p_1 = 1.127, \ p_2 = 18.006 \) and \( p_3 = 53.326 \).

\[
\text{Corr VS equivalent payoff } \left[ -\ln\left(\frac{S_T}{75}\right) + \frac{S_T}{75} - 1 \right] \mathbf{1}_{\left[\frac{75}{F_T}, \infty\right)}\left(S_T/F_T\right) \text{ (blue line)} \text{ and the consequent sub-hedging portfolio (black line) and super-hedging portfolio (red line).}
\]
**Numerical example for** $n = 3$ (cont):

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![Graph showing Gamma Swap equivalent payoff](image)

Gamma Swap equivalent payoff $\frac{S_T}{F_T} \ln \left( \frac{S_T}{F_T} \right) - \frac{S_T}{F_T}$ (blue line) and the consequent sub-hedging portfolio (black line). The portfolio is given by $\pi_1^\dagger = 0.00772$, $\pi_2^\dagger = 0.00571$, $\pi_3^\dagger = -0.00225$, $\phi^\dagger = 0$ and $\psi^\dagger = -0.929$. 
**Market Example:** Variance swaps on S&P500 Index

<table>
<thead>
<tr>
<th>Term</th>
<th>Quote date</th>
<th>VS quote</th>
<th>LB</th>
<th>No. of puts</th>
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</thead>
<tbody>
<tr>
<td>2M</td>
<td>20/04/2008</td>
<td>21.78</td>
<td>18.73</td>
<td>58</td>
</tr>
<tr>
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*VS quote source: Peter Carr & Liuren Wu*

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**Closing remarks**

- We derive model-independent no-arbitrage bounds, and associated super/sub-hedges, on prices of a European option with convex payoff, given market prices of finite set of co-maturing puts. This is equivalent to robust pricing and hedging of weighted variance swaps, assuming continuity of paths.

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THANK YOU
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Deduce unique prices and hedges.

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Unified approach
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Main results
Upper bound
Lower bound

Motivating questions and FTAP with market input
Weighted variance swaps
Robust pricing and hedging of options with convex payoffs

References:


2. A.M.G. Cox, J.O., *Robust pricing and hedging of double no-touch options*, forthcoming in *Finance and Stochastics*