Skew Modeling

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I. Generalities
Market Skews

Dominating fact since 1987 crash: strong negative skew on Equity Markets

Not a general phenomenon

Gold:  

FX:  

We focus on Equity Markets
Skews

- Volatility Skew: slope of implied volatility as a function of Strike
- Link with Skewness (asymmetry) of the Risk Neutral density function $\varphi$ ?

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Why Volatility Skews?

- Market prices governed by
  - a) Anticipated dynamics (future behavior of volatility or jumps)
  - b) Supply and Demand

- To “arbitrage” European options, estimate a) to capture risk premium b)
- To “arbitrage” (or correctly price) exotics, find Risk Neutral dynamics calibrated to the market

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Modeling Uncertainty

Main ingredients for spot modeling

• Many small shocks: Brownian Motion (continuous prices)

• A few big shocks: Poisson process (jumps)
2 mechanisms to produce Skews (1)

- To obtain downward sloping implied volatilities
  - a) Negative link between prices and volatility
    - Deterministic dependency (Local Volatility Model)
    - Or negative correlation (Stochastic volatility Model)
  - b) Downward jumps
2 mechanisms to produce Skews (2)

- a) Negative link between prices and volatility

- b) Downward jumps
Leverage and Jumps
Dissociating Jump & Leverage effects

\[ x = S_{t_1} - S_{t_0} \quad \text{and} \quad y = S_{t_2} - S_{t_1} \]

- **Variance**: \((x+y)^2 = x^2 + 2xy + y^2\)
  - Option prices \(\Delta\) Hedge
  - FWD variance

- **Skewness**: \((x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\)
  - Option prices \(\Delta\) Hedge
  - Leverage
  - FWD skewness
Dissociating Jump & Leverage effects

Define a time window to calculate effects from jumps and Leverage. For example, take close prices for 3 months

- Jump: \[ \sum_i \left( \delta S_{t_i} \right)^3 \]

- Leverage: \[ \sum_i \left( S_{t_i} - S_{t_1} \right) \left( \delta S_{t_i} \right)^2 \]
Dissociating Jump & Leverage effects
Dissociating Jump & Leverage effects
Break Even Volatilities
Theoretical Skew from Prices

Problem: How to compute option prices on an underlying without options?

For instance: compute 3 month 5% OTM Call from price history only.

1) Discounted average of the historical Intrinsic Values.
   Bad: depends on bull/bear, no call/put parity.

2) Generate paths by sampling 1 day return recentered histogram.
   Problem: CLT => converges quickly to same volatility for all strike/maturity; breaks autocorrelation and vol/spot dependency.
3) Discounted average of the Intrinsic Value from recentered 3 month histogram.

4) $\Delta$-Hedging: compute the implied volatility which makes the $\Delta$-hedging a fair game.
Theoretical Skew from historical prices (3)

How to get a theoretical Skew just from spot price history?

Example:

3 month daily data

1 strike \( K = k S_{T_1} \)

- a) price and delta hedge for a given \( \sigma \) within Black-Scholes model
- b) compute the associated final Profit & Loss: \( PL(\sigma) \)
- c) solve for \( \sigma(k) / PL(\sigma(k)) = 0 \)
- d) repeat a) b) c) for general time period and average
- e) repeat a) b) c) and d) to get the “theoretical Skew”
Theoretical Skew from historical prices (4)

S&P500 12/22/1999

Options prices

Break even vol
Theoretical Skew from historical prices (4)
Theoretical Skew from historical prices (4)

US Dollar in Yen 08/01/1995
Theoretical Skew from historical prices (4)

Gold 9/19/2001
Barriers as FWD Skew trades
Beyond initial vol surface fitting

• Need to have proper dynamics of implied volatility
  – Future skews determine the price of Barriers and OTM Cliquets
  – Moves of the ATM implied vol determine the $\Delta$ of European options

• Calibrating to the current vol surface do not impose these dynamics
Barrier Static Hedging

Down & Out Call Strike K, Barrier L, r=0:

- With BS: \( DOC_{K,L} = C_K - \frac{K}{L} P_{L^2/K} \)

If \( S_t = L \), unwind hedge, at 0 cost

If not touched, IV’s are equal

- With normal model

\[
DOC_{K,L} = C_K - P_{2L-K}
\]

\[
dS = \sigma dW
\]
Static Hedging: Model Dominance

- Back to $DOC_{K,L}$

- An assumption as the skew at $L$ corresponds to an affine model

  \[ dS = (aS + b) dW \]  
  (displaced LN)

- $DOC_{K,L}$ priced as in BS with shifted $K$ and $L$ gives new hedging PF which is $>0$ when $L$ is touched if Skew assumption is conservative
Skew Adjusted Barrier Hedges

\[ dS = (aS + b)dW \]

\[ \text{DOC}_{K,L} \leftrightarrow C_K - \frac{aK + b}{aL + b} P \left( \frac{aL^2 + b(2L - K)}{aK+b} \right) \]

\[ \text{UOC}_{K,L} \leftrightarrow C_K - (L - K) \left( 2 \text{Dig}_L + \frac{a}{aL + b} C_L \right) - \frac{aK + b}{aL + b} C \left( \frac{aL^2 + b(2L - K)}{aK+b} \right) \]
Local Volatility Model
One Single Model

• We know that a model with $dS = \sigma(S,t)dW$ would generate smiles.
  – Can we find $\sigma(S,t)$ which fits market smiles?
  – Are there several solutions?

ANSWER: One and only one way to do it.
The Risk-Neutral Solution

But if drift imposed (by risk-neutrality), uniqueness of the solution

Diffusions

Risk Neutral Processes

Compatible with Smile

sought diffusion (obtained by integrating twice Fokker-Planck equation)
Forward Equations (1)

• BWD Equation: 
  price of one option $C(K_0, T_0)$ for different $(S, t)$

• FWD Equation: 
  price of all options $C(K, T)$ for current $(S_0, t_0)$

• Advantage of FWD equation:
  – If local volatilities known, fast computation of implied volatility surface,
  – If current implied volatility surface known, extraction of local volatilities,
  – Understanding of forward volatilities and how to lock them.
Forward Equations (2)

• Several ways to obtain them:
  – Fokker-Planck equation:
    • Integrate twice Kolmogorov Forward Equation
  – Tanaka formula:
    • Expectation of local time
  – Replication
    • Replication portfolio gives a much more financial insight
Fokker-Planck

- If \( dx = b(x, t)dW \)

- Fokker-Planck Equation: \( \frac{\partial \varphi}{\partial t} = \frac{1}{2} \frac{\partial^2 (b^2 \varphi)}{\partial x^2} \)

- Where \( \varphi \) is the Risk Neutral density. As \( \varphi = \frac{\partial^2 C}{\partial K^2} \)

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial C}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial^2 C}{\partial K^2} \right) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( b^2 \frac{\partial^2 C}{\partial K^2} \right)
\]

- Integrating twice w.r.t. \( x \): \( \frac{\partial C}{\partial t} = \frac{b^2}{2} \frac{\partial^2 C}{\partial K^2} \)
Volatility Expansion

- K,T fixed. $C_0$ price with LVM: $\sigma_0(S_t, t): dS_t = \sigma_0(S_t, t)dW_t$
- Real dynamics: $dS_t = \sigma_t dW_t$

- Ito $\sigma^+(S_T - K) = C_0(S_0, 0) + \int_0^T \frac{\partial C_0}{\partial S} dS + \frac{1}{2} \int_0^T \frac{\partial^2 C_0}{\partial S^2} (\sigma_t^2 - \sigma_0^2(S_t, t)) dt$

- Taking expectation:
  $C(S_0, 0) = C_0(S_0, 0) + \frac{1}{2} \iint \Gamma_0(S, t)(E[\sigma_t^2|S_t = S] - \sigma_0^2(S, t)) \varphi(S, t) dS dt$

- Equality for all (K,T) $\Leftrightarrow$ $E[\sigma_t^2|S_t = S] = \sigma_0^2(S, t)$
Summary of LVM Properties

\[ \Sigma_0 \] is the initial volatility surface

- \( \sigma(S,t) \) compatible with \( \Sigma_0 \leftrightarrow \sigma = \text{local vol} \)
- \( \sigma(\omega) \) compatible with \( \Sigma_0 \leftrightarrow E[\sigma^2|S_T=K] = (\text{local vol})^2 \)
- \( \hat{\sigma}_{k,T} \) deterministic function of \( (S,t) \) (if no jumps)
  \( \leftrightarrow \) future smile = FWD smile from local vol
Stochastic Volatility Models
Heston Model

\[
\begin{align*}
\frac{dS}{S} &= \mu \, dt + \sqrt{v} \, dW \\
v &= \kappa (v_\infty - v) \, dt + \eta \sqrt{v} \, dZ \quad \langle dW, dZ \rangle = \rho \, dt
\end{align*}
\]

Solved by Fourier transform:

\[
x \equiv \ln \frac{FWD}{K} \quad \tau = T - t
\]

\[
C_{K,T}(x, v, \tau) = e^x P_1(x, v, \tau) - P_0(x, v, \tau)
\]
Role of parameters

• Correlation gives the short term skew
• Mean reversion level determines the long term value of volatility
• Mean reversion strength
  – Determine the term structure of volatility
  – Dampens the skew for longer maturities
• Volvol gives convexity to implied vol
• Functional dependency on S has a similar effect to correlation
Impact of $\sigma_\infty$
Spot dependency

2 ways to generate skew in a stochastic vol model

1) $\sigma_t = x_t f(S,t), \rho(W,Z) = 0$

2) $\sigma, \rho(W,Z) \neq 0$

- Mostly equivalent: similar $(S_t, \sigma_t)$ patterns, similar future evolutions
- 1) more flexible (and arbitrary!) than 2)
- For short horizons: stoch vol model $\Leftrightarrow$ local vol model
  + independent noise on vol.
SABR model

- F: Forward price

\[ dF = F^\beta \sigma_t dW \]

\[ \frac{d\sigma}{\sigma} = \alpha \ dZ \]

- With correlation \( \rho \)
Impact of beta

maturity

strike
Impact of rho and vol vol
Impact of beta and volvol
Impact of volvol
Smile Dynamics
Smile dynamics: Local Vol Model (1)

- Consider, for one maturity, the smiles associated to 3 initial spot values

Skew case

- ATM short term implied follows the local vols
- Similar skews
Smile dynamics: Local Vol Model (2)

- Pure Smile case

- ATM short term implied follows the local vols
- Skew can change sign
Smile dynamics: Stoch Vol Model (1)

Skew case \( r < 0 \)

- ATM short term implied still follows the local vols

\[
\left( E \left[ \sigma^2_T \mid S_T = K \right] = \sigma^2(K,T) \right)
\]

- Similar skews as local vol model for short horizons
- Common mistake when computing the smile for another spot: just change \( S_0 \) forgetting the conditioning on \( \sigma \):
  if \( S : S_0 \rightarrow S^+ \) where is the new \( \sigma \) ?
Smile dynamics: Stoch Vol Model (2)

- Pure smile case ($r=0$)

- ATM short term implied follows the local vols
- Future skews quite flat, different from local vol model
- Again, do not forget conditioning of vol by $S$
Smile dynamics: Jump Model

Skew case

- ATM short term implied constant (does not follow the local vols)
- Constant skew
- Sticky Delta model
Smile dynamics: Jump Model

Pure smile case

- ATM short term implied constant (does not follow the local vols)
- Constant skew
- Sticky Delta model
Weighting scheme imposes some dynamics of the smile for a move of the spot:
For a given strike $K$,
$$ S \uparrow \Rightarrow \bar{\sigma}_K \downarrow $$
(we average lower volatilities)
Smile today (Spot $S_t$) & Smile tomorrow (Spot $S_{t+dt}$) in sticky strike model
Smile tomorrow (Spot $S_{t+dt}$) if $\sigma_{\text{ATM}} = \text{constant}$
Smile tomorrow (Spot $S_{t+dt}$) in the smile model
Volatility Dynamics of different models

- Local Volatility Model gives future short term skews that are very flat and Call $\Delta$ lesser than Black-Scholes.
- More realistic future Skews with:
  - Jumps
  - Stochastic volatility with correlation and mean-reversion
- To change the ATM vol sensitivity to Spot:
  - Stochastic volatility does not help much
  - Jumps are required
ATM volatility behavior
In the absence of jump:

\[
\text{model fits market } \iff \forall K,T \quad E[\sigma_T^2|S_T = K] = \sigma_{loc}^2(K,T)
\]

This constrains

a) the sensitivity of the ATM short term volatility wrt $S$;

b) the average level of the volatility conditioned to $S_T = K$.

a) tells that the sensitivity and the hedge ratio of vanillas depend on the calibration to the vanilla, not on local volatility/ stochastic volatility.

To change them, jumps are needed.

But b) does not say anything on the conditional forward skews.
Sensitivity of ATM volatility / S

At t, short term ATM implied volatility $\sim \sigma_t$.

As $\sigma_t$ is random, the sensitivity $\frac{\partial \sigma_t}{\partial S}$ is defined only in average:

$$E_t\left[\sigma_{t+\delta t}^2 - \sigma_t^2 \mid S_{\delta t} = S_t + \delta S\right] = \sigma_{loc}^2 (S_t + \delta S, t + \delta t) - \sigma_{loc}^2 (S_t, t) \approx \frac{\partial \sigma_{loc}^2 (S, t)}{\partial S} \cdot \delta S$$

In average, $\sigma_{ATM}^2$ follows $\sigma_{loc}^2$.

Optimal hedge of vanilla under calibrated stochastic volatility corresponds to perfect hedge ratio under LVM.
Market Model of Implied Volatility

- Implied volatilities are directly observable
- Can we model directly their dynamics? \((r = 0)\)

\[
\begin{align*}
\frac{dS}{S} &= \sigma dW_1 \\
\frac{d\hat{\sigma}}{\hat{\sigma}} &= \alpha dt + u_1 dW_1 + u_2 dW_2
\end{align*}
\]

where \(\hat{\sigma}\) is the implied volatility of a given \(C_{K,T}\)

- Condition on \(\hat{\sigma}\) dynamics?
Drift Condition

- Apply Ito’s lemma to $C(S, \hat{\sigma}, t)$
- Cancel the drift term
- Rewrite derivatives of $C(S, \hat{\sigma}, t)$

gives the condition that the drift $\alpha$ of $\frac{d\hat{\sigma}}{\hat{\sigma}}$ must satisfy.

For short T, we get the Short Skew Condition (SSC):

$$\hat{\sigma}^2 = \left( \sigma + u_1 \ln\left(\frac{K}{S}\right) \right)^2 + \left( u_2 \ln\left(\frac{K}{S}\right) \right)^2$$

close to the money: $\hat{\sigma}^2 \sim \sigma + u_1 \ln\left(\frac{K}{S}\right)$

$\Rightarrow$ Skew determines $u_1$
Optimal hedge ratio $\Delta^H$

- $C(S, \sigma, t)$: BS Price at $t$ of Call option with strike $K$, maturity $T$, implied vol $\sigma$
- Ito: $dC(S, \sigma, t) = 0dt + C_S dS + C_\sigma d\sigma$
- Optimal hedge minimizes P&L variance:

$$\Delta^H = \frac{dC . dS}{(dS)^2} = C_S + C_\sigma \frac{d\sigma . dS}{(dS)^2}$$

- BS Delta
- BS Vega
- Implied Vol sensitivity
Optimal hedge ratio $\Delta^H$ II

$$\Delta^H = C_s + C_\hat{\sigma} \frac{d\hat{\sigma}.dS}{(dS)^2}$$

With

$$\begin{align*}
\frac{dS}{S} &= \sigma dW_1 \\
\frac{d\hat{\sigma}}{\hat{\sigma}} &= \alpha dt + u_1 dW_1 + u_2 dW_2
\end{align*}$$

$$\frac{d\hat{\sigma}.dS}{(dS)^2} = \frac{u_1 \sigma S}{(\sigma S)^2} \frac{(dW_1)^2}{(dW_1)^2} = \frac{u_1 \hat{\sigma}}{\sigma S}$$

$\Rightarrow$ Skew determines $u_1$, which determines $\Delta^H$
Smile Arbitrage
Deterministic future smiles

It is not possible to prescribe just any future smile.

If deterministic, one must have

\[ C_{K,T_2}(S_0,t_0) = \int \varphi(S_0,t_0,S,T_1) C_{K,T_2}(S,T_1) dS \]

Not satisfied in general.
Det. Fut. smiles & no jumps

$\Rightarrow = \text{FWD smile}$

If $\exists(S, t, K, T) / V_{K,T}(S, t) \neq \bar{\sigma}^2(K, T) \equiv \lim_{\delta K \to 0, \delta T \to 0} \sigma^2_{imp}(K, T, K + \delta K, T + \delta T)$

stripped from Smile S.t

Then, there exists a 2 step arbitrage:

Define

$$PL_t = \left(\bar{\sigma}^2(K, T) - V_{K,T}(S, t)\right) \frac{\partial^2 C}{\partial K^2}(S, t, K, T)$$

At $t_0$: Sell $PL_t \cdot \left(Dig_{S-\varepsilon, t} - Dig_{S+\varepsilon, t}\right)$

At $t$: if $S_t \in [S - \varepsilon, S + \varepsilon]$ buy $\frac{2}{K^2}CS_{K,T}$, sell $\bar{\sigma}^2(K, T)\delta_{K,T}$

gives a premium = PL$t$ at $t$, no loss at $T$

Conclusion: $V_{K,T}(S, t)$ independent of $(S, t) = V_{K,T}(S_0, t_0) = \sigma^2(K, T)$

from initial smile
Consequence of det. future smiles

• Sticky Strike assumption: Each \((K,T)\) has a fixed \(\sigma_{\text{impl}}(K,T)\) independent of \((S,t)\)

• Sticky Delta assumption: \(\sigma_{\text{impl}}(K,T)\) depends only on moneyness and residual maturity

• In the absence of jumps,
  – Sticky Strike is arbitrageable
  – Sticky \(\Delta\) is (even more) arbitrageable
Example of arbitrage with Sticky Strike

Each CK,T lives in its Black-Scholes ($\sigma_{impl}(K,T)$) world

$C_1 \equiv C_{K_1,T_1}$  $C_2 \equiv C_{K_2,T_2}$  assume $\sigma_1 > \sigma_2$

P&L of Delta hedge position over $dt$:

\[ \delta PL \left( C_1 \right) = \frac{1}{2} \left( (\delta S)^2 - \sigma_1 S^2 \delta t \right) \Gamma_1 \]

\[ \delta PL \left( C_2 \right) = \frac{1}{2} \left( (\delta S)^2 - \sigma_2 S^2 \delta t \right) \Gamma_2 \]

\[ \delta PL \left( \Gamma_1 C_2 - \Gamma_2 C_1 \right) = \frac{\Gamma_1 \Gamma_2}{2} S^2 \left( \sigma_1^2 - \sigma_2^2 \right) \delta t > 0 \]

(no $\Gamma$, free $\Theta$)

⚠️ If no jump
Arbitrage with Sticky Delta

- In the absence of jumps, Sticky-K is arbitrageable and Sticky-Δ even more so.
- However, it seems that quiet trending market (no jumps!) are Sticky-Δ.
  - In trending markets, buy Calls, sell Puts and Δ-hedge.

Example:

\[ PF \equiv C_{K_2} - P_{K_1} \]

\[ S \uparrow \longrightarrow \begin{cases} \sigma_1, \sigma_2 \uparrow \\ \text{Vega}_{K_2} > \text{Vega}_{K_1} \end{cases} \longrightarrow PF \]

\[ S \downarrow \longrightarrow \begin{cases} \sigma_1, \sigma_2 \downarrow \\ \text{Vega}_{K_2} < \text{Vega}_{K_1} \end{cases} \longrightarrow PF \]

Δ-hedged PF gains from S induced volatility moves.
Conclusion

• Both leverage and asymmetric jumps may generate skew but they generate different dynamics

• The Break Even Vols are a good guideline to identify risk premia

• The market skew contains a wealth of information and in the absence of jumps,
  – The spot correlated component of volatility
  – The average behavior of the ATM implied when the spot moves
  – The optimal hedge ratio of short dated vanilla
  – The price of options on RV

• If market vol dynamics differ from what current skew implies, statistical arbitrage