Portfolio Credit Derivatives
The state of affairs

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Agenda

1. CDSs, STCDOs, and all that
3. Base correlation skew and its implications
4. Models for the base skew
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References:


1. STCDOs, CDSs, and all that

- Consider $N$ firms with default dates $\tau_i$. Loss-given default for firm $i$ is $l_i$.

- Total loss on portfolio is $L(T) = \sum_i 1_{\tau_i < T} l_i$, with mean

$$E(L(T)) = E\left(\sum_i l_i 1_{\tau_i < T}\right) = \sum_i l_i \Pr(1_{\tau_i < T}).$$

- In a CDS (credit default swap), somebody insures somebody else against credit loss on a single firm, in return for a stream of fees

- CDSs are observable, allowing for easy computation of all (risk-neutral) default probabilities

- In an STCDO (single-tranche collateralized debt obligation), somebody insures somebody else against a tranche of losses on the entire $N$-name portfolio.

- Specifically, for a tranche covering $[x,y]$ the covered losses are only those in $[x,y]$. The protection buyer has a call spread on losses: cumulative paid losses on $[0,T]$ are

$$MAX(L(T) - x, 0) - MAX(L(T) - y, 0).$$

- A CDO-squared contract is an STCDO where the underlying securities are STCDOs, rather than CDSs.
2. Basic factor set-up. Standard model.

• To price call options on portfolio losses (and thereby price STCDOs and other portfolio derivatives), we need a model for portfolio losses.

• The key driver of this distribution is obviously default co-dependence: the more there is, the fatter the upper ("crash") tail will be, and the more valuable out-of-the-money options (=senior tranches) will be.

• Due to a) the high dimensionality (\(N\)) of the problem; b) the need to compute many (often 1000’s) of hedge numbers per trade; and c) the lack of observable “co-dependence” data, CDO models tend to be simplistic.

• The standard approach is to introduce a low-dimensional set of market drivers \(Z\), and then simply have all firms’ survival probabilities be some exogenously specified function of \(Z\):

\[
\Pr(\tau_i > t \mid Z = z) = f_i(t, z).
\]  

(*)

• (*) needs to be supplemented with a choice of the (possibly \(t\)-dependent) probability density \(\varphi(t, z)\). And we must enforce the constraints

\[
\Pr(\tau_i > T) = \int_{\mathbb{R}^d} f_i(t, z)\varphi(t, z)dz, \ i=1,\ldots,N
\]  

(**)  

where l.h.s. is prescribed by CDS market.
• Key trick: once we condition on $Z$, defaults become independent, allowing us to very quickly construct the loss distribution by standard recursion on convolution methods. See [1]-[2] for details.

• We integrate out the conditioning on $Z$ in a simple $d$-dimensional integral (which is low-dimensional by assumption).

• For speed reasons, it is really best if (**) can be done in closed form.

• One model where this is possible is the Gaussian copula, the de-facto market standard model. Let us just look at the simplest one-factor case with constant correlation $\rho$.

\[
1_{\tau_t \leq T} = 1_{X_1 \leq H_i(T)};
X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon_i, \ i = 1, \ldots, N
\]

• Here $Z$ is the standard Gaussian (1-D) and $\varepsilon_i$ are independent Gaussian residuals. It easily follows that

\[
f_i(t,z) = 1 - \Phi\left(\frac{H_i(t) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right);
\phi(t,z) = \phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}};
\int_{\mathbb{R}} f_i(t,z)\phi(t,z)dz = 1 - \Phi(H_i(t)).
\]
• The calibration functions $H_i(t)$ can easily be obtained in closed-form here. They – roughly – have the interpretation of default barriers, below which the asset return must fall before default takes place.

• The variable $Z$ is often considered the “market return”, with $\sqrt{\rho}$ being the market “beta”.

• The Gaussian copula is easy to use, but it has many known drawbacks:
  a) Does not constitute a proper dynamic model, so notions of ‘’hedging’’ are poorly defined
  b) Bad stationarity properties
  c) Neglects the random nature of recovery rates
  d) Does not match the market for standardized tranches (CDX, I-Traxx, etc.)
  e) …

• a) is pretty much par for the course for all factor-type models. But we can certainly attempt to fix b),c),d).
Base correlation skew and its implications

- From the market for tranches on quoted indices (I-Traxx, CDX, etc.), one can imply the empirical risk-neutral loss distribution

- This distribution has a fat upper tail, and low probability of generating zero losses:

\[ \text{Probability of Loss} \leq L \]

- Market standard is to represent the loss distribution through implied Gaussian copula correlations on tranches \([0,u]\), sometimes called base correlations. (A tranche covering \([x,y]\) can be computed as the difference between a \([0,y]\) and an \([0,x]\) tranche).
• Market is interpolating, extrapolating, and massaging the correlation skew when pricing regular CDO tranches

• Some care must be taken in this exercise – not all interpolation schemes are arbitrage-free (and not all base correlation skews are allowed). Also, there are arbitrage-free prices that have no implied base correlation skew

• Sometimes it appears that the market is treating base correlations (even when measured against fixed, rather than loss-adjusted, detachment levels) as fundamental invariants
• Recall, however, that base correlation is just one particular (and arbitrary) way to indicate the form of the market-implied risk-neutral loss distribution. There is not necessarily anything fundamental in play.

• We could equally well have used, say, implied Black-Scholes volatilities as a characterization of the form of the loss distribution:

• Other possible interpolation rules: implied loss intensity, implied odd’s ratio, etc., etc.

• The close proximity of 5-yr and 10-yr CDX base correlation skews is surprising, particularly given that they are mapped against absolute tranche levels. If we translate detachment levels into multiples of expected portfolio loss, the base correlation graphs look like this:
Extrapolation rules based on detachment relative to expected loss seems reasonable (and is a lot more model-friendly), but the market does not conform at the time being.

We must emphasize that the base correlation methodology is not a model, but only an interpolation mechanism: “wrong number in the wrong formula to get the right result”

This manifests itself in a number of ways. Most obvious in computation of spread hedges: how should base correlations move with spreads?

The common “rule” is to assume that base correlation is independent of spreads. But this can a) lead to arbitrages; and b) can produce weird hedges. Example:
(CDX, Summer 2005, senior tranche)
Models for the base skew.

Consider setting up a factor framework that can produce a base correlation skew. A number of approaches exist in the literature, with various degrees of economic intuition and ability to match the (rather extreme) smile in the market.

1. Mixing Correlations

- Basic idea: price tranches at several different flat correlations, ranging from high to low. Take average.

- Is very simple and can generate a skew – but not a very steep one. Will typically misprice mezzanine tranche quite significantly.

- Can be extended to include recoveries, which will steepen the smile (but not for fixed-recovery deals)

- Factor representation

\[ f_i(t, z) = 1 - \Phi \left( \frac{H_i(t) - \sqrt{z_2 z_1}}{\sqrt{1 - z_2}} \right) \]

where \( z_1 \) is Gaussian and \( z_2 \) is an independent random variable taking discrete values in \([0,1]\).
2. *Heterogenous Correlation Matrix*

- Basic idea: let correlations be heterogenous and functions of spread. Low spread names are associated with higher correlation factors than high-spread names.

- This generates a correlation skew, but again it tends to be too weak.

- Also, very strong spread/correlation dependence is needed, much beyond what can be supported empirically.

- Factor representation is same as constant-correlation Gaussian copula, but $\rho$ is replaced with $\rho_i$.

3. *Intensity mixtures*

- Here we set

$$f_i(z,t) = e^{-z h_i(t)}$$

which has nice interpretation of being a proportional perturbation of the hazard rate curve.

- Generally not tractable, except for the Gamma-distributed case where

$$\varphi(z) = \frac{z^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-z/\beta}.$$  

- Then
\[ \int_{\mathbb{R}} f(t, z) \varphi(z) dz = \frac{1}{(1 + \beta h(t))^\alpha} \]

- Fit is not particularly good with one factor; can be improved by using multiple independent Gamma distributions (at computational cost)

- The form for \( f \) is nice -- suggestions in the literature to imply the distribution of \( z \) non-parametrically. Unclear whether this inverse problem is stable and computationally feasible (there no tractability).

4. “Local” correlation (RFL model)

- One reasonably intuitive idea is to take the factor loading \( \rho \) in the Gaussian copula and make it a function of the systematic variable \( Z \).

- We can loosely think of this as making correlation a function of the “economy” or the business cycle.

- It is a well-documented phenomenon that correlation increases in a downturn, so there is some empirical meaning here.

- The idea is also convenient, because we can retain the shell of the factor-based Gaussian set-up with all its well-developed numerical machinery.
• Specifically, we write

\[ L(T) = \sum_{i=1}^{N} 1_{X_i \leq H_i} l_i, \quad X_i = a(Z)Z + \nu \epsilon_i + m \]

• Here \( Z \) and \( \epsilon_i \) are Gaussian, \( a(Z) \) is a function \( \mathbb{R} \rightarrow \mathbb{R} \), \( \nu \) and \( m \) are constants.

• Note: cousins of the model exist where the residuals are also function of \( Z \), as in \( X_i \equiv a(Z)Z + \sqrt{1 - a(Z)^2} \epsilon_i \)

• Can easily extend to vector-valued \( Z \) and to firm-specific \( a(Z) \) (see [4]).

• Factor representation of RFL model is simple

\[ f_i(t, z) = 1 - \Phi \left( \frac{H_i(t) - a(Z)Z - m}{\nu} \right) \]

• How to pick \( a \) function? One could try to non-parametrically “imply” it from the correlation skew (an inverse problem), but this is likely unstable and temperamental.

• Likely better to impose \( a \) from the outside and fit a few parameters. In this case, we want \( a \) to be tractable and monotonically declining (such that high \( Z \) = good “economy” implies low correlation, and vice versa).
• There are many tractable specifications – see [4]. For instance, any piecewise flat function leads to closed-form expressions for the density of the $X_i$, closed-form solutions for default probabilities.

• Here are some (casual) fits for a simple three-step function for $a(Z)$. Fit is to 5-year CDX and takes about 10 seconds to run:

Fall 2004
The current skew is very close to arbitrary bounds

5. Other parametric distributions.

- Rather than alter the factor loading function in the Gaussian copula, we could set the factor-loading to a constant and alter the distribution of the common factor

- This corresponds to keeping the $f_i(t, z)$ functions same as for Gaussian copula, but changing the distribution of $z$. 
• The NIG (Normal Inverse Gaussian) is particularly useful, as it allows for a (complicated) closed-form solution to the $X_i$, making calibration relatively easy.


• A less “smooth” approach introduces default dependencies by introducing systemic crashes (Marshal-Olkin copula) or by using jump-diffusion processes for the default driver $X_i$. See [5] for the formulation of these in the factor framework.

• Most smooth models (like RFL) can be appended with some type of systemic crash event, to better handle the extreme upper loss tail.

• In fact, many of the models described so far can be combined while retaining tractability. See [5].
6. Model Issues

- Why did we bother setting up a model for the base skew in the first place? For STCDOs at least, once we have the base correlation skew, an interpolator is all that is needed.

- First, a model ensures arbitrage-free interpolation and extrapolation – this is not always the case with market-standard rules of thumb.

- A model is required to price non-standard instruments, such as CDO^2.

- A model is needed to make sense of hedges, and how base correlation moves with spreads.

- But factor-models are not ideal for any of these things: a) nobody knows if factor model parameters are “fundamental” enough to allow for extrapolation to non-index tranches (different maturities, composition, etc.); b) calibration to CDO^2 is difficult; and c) hedges are poorly defined in factor models

- Let’s look at a few of these issues.
Hedging.

- In a factor model, spread of firm $i$ can move for two fundamental reasons: a) the function $f_i$ changes; or b) the distribution $\varphi$ changes. First is an idiosyncratic shift; second is a systemic shift affecting multiple firms simultaneously.

- To give a very simple example, consider a MO copula where each name is characterized by a default intensity of the form

$$\lambda_i = \overline{\lambda}_i + \lambda$$

where $\overline{\lambda}_i$ is an idiosyncratic intensity and $\lambda$ is the intensity of a “kill-all” event that will cause all firms to simultaneously default.

- In this perfectly stationary model, a spread can increase for two reasons: i) $\overline{\lambda}_i$ goes up; or ii) $\lambda$ goes up.

- The two events lead to very different moves of the base correlation smile:
• When $\lambda$ goes up, the base correlation smile increases; when $\overline{\lambda}_t$ goes up (across the board), the base correlation smile decrease. Pretty intuitive here.

• So the effective leverage of a tranche is not well-defined – depends on the type of shift.

• Same in RFL model: if a spread move is assumed all “idiosyncratic” (reason a)), the correlation skew shifts down when spreads increase.

• Interpretation: increasing spreads make any fixed tranches less senior, hence we slide down on the correlation curve. Or: increasing the idiosyncratic component of the spread adds noise -> less correlation.

• But if the spreads move because the systematic factor Z moved down (reason b)), then, by construction, the base correlations move up.
• The model suggests that one needs to hedge both against idiosyncratic and systematic shifts, but it’s not very specific about how to do this. If we want a single, unique hedge, we may have to define a rule ourselves (as “sticky-delta”, “sticky-strike” in equity markets)

• Such a rule could be model-based.

**Maturity Extrapolation**

• The Gaussian copula is considered to have bad stationarity properties, so let us quickly go back to our “kill-all” model:

![Graph showing the relationship between Detachment Level and Base Correlation across different maturities.](graph)

• So: a marked decrease in base correlation for increasing maturity.

• The RFL model is very similar (in fact, it is a lot more stationary than a regular Gaussian copula)
But unfortunately, the market does not really comply (as we have seen earlier): the maturity dependence of the base correlation smile is minor. As a consequence, it is hard to fit models to, say, 10-year market.

Another way of saying this: model parameters are pretty maturity dependent, which is not ideal.

On the other hand, the factor framework allows for time-dependent parameters, so one could attempt to introduce enough time-dependence to match market. May also be realistic: in a dynamic sense, an RFL model should probably have \( a(Z) = a(Z\sqrt{t}) \). See [5].

**Non-vanilla CDOs – CDO^2**

For consistent pricing of CDO^2, model should correctly reproduce skews on tranched sub-baskets

RFL model can be calibrated to tranche prices on each sub-basket separately

By using RFL model with name-specific parameters can arrange to match sub-basket tranche prices whilst preserving consistency on overlaps

Procedure is not unique, but gives intuitive results.

By the way: CDO^2 product must be done by Monte Carlo, but with the nice results of [6], we can do this quite efficiently, even with computation of hedges.
7. Conclusions.

- There are a number of models which can explain the presence of a base correlation smile. The RFL model is one of the better ones.

- Still, the model has significant drawbacks, including difficulty of matching observed term structures of correlations, and non-unique hedges.

- At this point, however, there are no models (that I know of) that give crisp, theoretically well-defined deltas while being computationally tractable and intuitive to traders.

- The practical value currently added by models is, sadly, pretty low, at least when it comes to basic CDO hedging.

- Still a lot of room left for new models. They will benefit from empirical observations in the future.

- We should keep in mind, that 30 years after Black-Scholes, traders still use this (flawed) model along with lots of rules to price and hedge vanilla options. Perhaps we are stuck with the Gaussian copula, too.

- Interesting (and difficult) questions remain in terms of extrapolation in portfolio composition and maturity.