Attilio Meucci

Managing Diversification
A. MEUCCI - Managing Diversification

COMMON MEASURES OF DIVERSIFICATION

DIVERSIFICATION DISTRIBUTION

MEAN-DIVERSIFICATION FRONTIER

CONDITIONAL ANALYSIS

REFERENCES
Common Measures of Diversification

\[ R_w \equiv w' \mathbf{R} \]

- Portfolio return
- Returns of securities (stocks, bonds, options, structured products, ...)
- Portfolio weights
Common Measures of Diversification

\[ R_w \equiv w' \mathbf{R} \]

- **weight-based definitions**
  
  \[ D_{Her} \equiv 1 - w'w \]

**portfolio weights**
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Common Measures of Diversification

\[ R_w \equiv w' \mathbf{R} \]

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distribution

portfolio weights
- positive
- sum to one
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Common Measures of Diversification

\[ R_w \equiv w' R. \]

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\[ D_{Her} \equiv 1 - w' w. \]

\[ D_{BP} \equiv - \sum_{n=1}^{N} w_n \ln(w_n). \]  

portfolio weights

- positive
- sum to one
**A. MEUCCI - Managing Diversification**  Common Measures of Diversification

\[ R_w \equiv w'R. \]

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\[ D^{(\gamma)}_{HK} \equiv - \left( \sum_{n=1}^{N} w_n^\gamma \right)^{\frac{1}{\gamma-1}}. \]

**Portfolio weights**

- positive
- sum to one

**Diagram**

- Illustration of portfolio weights
- Security number

**Notes**

- Distribution
- Entropy
Common Measures of Diversification

weight-based definitions

\[ R_w \equiv w' R. \]

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Common Measures of Diversification

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- Risk-based definitions

\[ D_{IP} \equiv 1 - w'Cw \]

Returns correlation matrix
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- risk-based definitions

\[ D_{IP} \equiv 1 - w' \mathbf{C} w, \]
\[ D_{Diff} \equiv \sigma' w - \sqrt{w' \Sigma w}. \]

returns standard deviations
returns covariance matrix
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\[ R_w \equiv w' R. \]

- weight-based definitions

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- risk-based definitions

\[ D_{IP} \equiv 1 - w' C w. \]

\[ D_{Diff} \equiv \sigma' w - \sqrt{w' \Sigma w}. \]

- factor-based definition

\[ R_n \equiv \sum_{k=1}^{K} \beta_{n,k} F_k + \epsilon_n \]

\[ D_{IS} \equiv 1 - \frac{\text{Var} \{ R_\epsilon \}}{\text{Var} \{ R_w \}} \]

portfolio return due to “idiosyncratic”
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\[ R_w \equiv w'R. \]

- weight-based definitions
  
  \[ D_{Her} \equiv 1 - w'w. \]
  
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- risk-based definitions
  
  \[ D_{IP} \equiv 1 - w'Cw. \]
  
  \[ D_{Diff} \equiv \sigma'w - \sqrt{w'\Sigma w}. \]

These definitions apply in specific circumstances and or under restrictive hypotheses.
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\[ R_w \equiv w' R. \]
Example: portfolio of two securities

- one bond  \( w_1 = 50\% \)  \( \text{Var}\{R_1\} = (1\%)^2 \)
- one stock  \( w_2 = 50\% \)  \( \text{Var}\{R_2\} = (30\%)^2 \)

if correlations = 0

\[ R_w \equiv w' \mathbf{R}. \]
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Example: portfolio of two securities

- one bond  \( w_1 = 50\% \)  \( Var\{R_1\} = (1\%)^2 \)
- one stock  \( w_2 = 50\% \)  \( Var\{R_2\} = (30\%)^2 \)

if correlations = 0

\[
R_w \equiv w^\prime \mathbf{R}.
\]

\[
Var\{R_w\} \equiv \sum_{n=1}^{N} Var\{w_n R_n\}
\]

weights highly diversified  

risk highly concentrated
if correlations ≠ 0

\[ R_w \equiv w' \mathbf{R} \]

\[ \text{Var} \{ R_w \} \neq \sum_{n=1}^{N} \text{Var} \{ w_n R_n \} \]
Diversification Distribution

\[ R_w \equiv w' \mathbf{R} \]

\[ \text{Var} \{ R_w \} \neq \sum_{n=1}^{N} \text{Var} \{ w_n R_n \} \]

if correlations = 0

Example: portfolio of two government bonds in same duration bucket

Bond 1 \( w_1 = 50\% \) \( \text{Var} \{ R_1 \} = (1\%)^2 \)

Bond 2 \( w_2 = 50\% \) \( \text{Var} \{ R_2 \} = (1\%)^2 \)
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Diversification Distribution

\[ R_w \equiv w' \mathbf{R} \]

\[ \text{Var} \{ R_w \} \neq \sum_{n=1}^{N} \text{Var} \{ w_n R_n \} \]

Example: portfolio of two government bonds in same duration bucket

<table>
<thead>
<tr>
<th>Bond 1</th>
<th>( w_1 = 50% )</th>
<th>( \text{Var} { R_1 } = (1%)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 2</td>
<td>( w_2 = 50% )</td>
<td>( \text{Var} { R_2 } = (1%)^2 )</td>
</tr>
</tbody>
</table>

Weighs highly diversified

Volatility homogeneous

High concentration due to correlations: full exposure to first principal component
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\[ R_w \equiv w' \Sigma \]

\[ \Sigma \equiv \text{Cov}\{R\} \]

\[ \Sigma \equiv E \Lambda E' \]

\[ E \equiv (e_1, \ldots, e_N) \]

\[ \Lambda \equiv \text{diag}(\lambda_1^2, \ldots, \lambda_N^2) \]

\[ \lambda_n^2 \equiv \text{Var}\{e_n'R\} \]

PCA

eigenvectors

principal portfolios

eigenvalues

principal variances

principal portfolio 1

principal portfolio 2

\[ R_2 \]

\[ R_1 \]
A. MEUCCI - Managing Diversification  Diversification Distribution

\[ R_w \equiv w' \Sigma \]

\[ \Sigma \equiv \text{Cov}\{R\} \]

\[ \Sigma \equiv \mathbf{E} \Lambda \mathbf{E}' \]

\[ \tilde{R} \equiv \mathbf{E}^{-1} R \]

return of principal portfolios
\[
R_w \equiv w' \mathbf{R}.
\]

\[
\Sigma \equiv \text{Cov}\{\mathbf{R}\}
\]

\[
\Sigma \equiv \mathbf{E} \Lambda \mathbf{E}'
\]

\[\tilde{\mathbf{R}} \equiv \mathbf{E}^{-1} \mathbf{R}\]  
return of principal portfolios

\[\tilde{w} \equiv \mathbf{E}^{-1} w\]  
weights of original portfolio on principal portfolios
\[ R_w \equiv w' \mathbf{R} \]

\[ \Sigma \equiv \text{Cov}\{\mathbf{R}\} \]

\[ \Sigma \equiv \mathbf{E} \Lambda \mathbf{E}' \]

\[ \mathbf{R} \equiv \mathbf{E}^{-1} \mathbf{R} \quad \text{return of principal portfolios} \]

\[ \mathbf{w} \equiv \mathbf{E}^{-1} \mathbf{w} \quad \text{weights of original portfolio on principal portfolios} \]

\[ R_w \equiv \mathbf{w}' \mathbf{R} \]
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Diversification Distribution

\[ R_w \equiv w'\mathbf{R} \]

\[ \text{variance concentration curve} \]

\[ \text{total variance} \]

\[ \tilde{\mathbf{R}} \equiv \mathbf{E}^{-1}\mathbf{R} \]

\[ \tilde{\mathbf{w}} \equiv \mathbf{E}^{-1}w \]

\[ v_n \equiv \tilde{w}_n^2\lambda_n \]

\[ \text{return of principal portfolios} \]

\[ \text{weights of original portfolio on principal portfolios} \]

\[ \text{variance concentration curve} \]

\[ \text{contribution to original portfolio variance from n-th principal portfolio:} \]

\[ \text{Var} \{R_w\} \equiv \sum_{n=1}^{N} v_n \]
Example: portfolio of two government bonds in same duration bucket

Bond 1 \( w_1 = 50\% \)

Bond 2 \( w_2 = 50\% \)

\[ \text{Var} \{ R_1 \} = (1\%)^2 \]

\[ \text{Var} \{ R_2 \} = (1\%)^2 \]

\( \tilde{R} \equiv E^{-1}R \)

\( \tilde{\omega} \equiv E^{-1}w \)

\( v_n \equiv \tilde{w}_n^2 \lambda_n \)

\[ \text{Var} \{ R_w \} \equiv \sum_{n=1}^{N} v_n \]
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\[ R_w \equiv w' \tilde{R}. \]

- **total volatility**
- **volatility concentration curve**

\[ \tilde{R} \equiv E^{-1}R \]

- return of principal portfolios

\[ \tilde{w} \equiv E^{-1}w. \]

- weights of original portfolio on principal portfolios

\[ v_n \equiv \tilde{w}_n^2 \lambda_n^2; \]

- variance concentration curve

\[ s_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{sd \{R_w\}} \]

- volatility concentration curve

contribution to original portfolio volatility from n-th principal portfolio: “hot spots”
\( R_w \equiv w' \tilde{R} \)

**Diversification Distribution**

\[\begin{align*}
\tilde{R} &\equiv E^{-1}R \\
\tilde{w} &\equiv E^{-1}w,
\end{align*}\]

- return of principal portfolios
- weights of original portfolio on principal portfolios

\[\begin{align*}
v_n &\equiv \tilde{w}_n^2 \lambda_n^2 \\
\sigma_n &\equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Sd} \{R_w\}} \\
p_n &\equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{R_w\}}
\end{align*}\]

- variance concentration curve
- volatility concentration curve
- diversification distribution
- contribution to original portfolio r-square from n-th principal portfolio
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\[ R_w \equiv w' \tilde{R}. \]

\[ \tilde{R} \equiv E^{-1} R \]

\[ \tilde{w} \equiv E^{-1} w, \]

return of principal portfolios

weights of original portfolio on principal portfolios

\[ v_n \equiv \tilde{w}_n^2 \lambda_n^2 \]

variance concentration curve

\[ s_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Sd} \{ R_w \}} \]

volatility concentration curve

\[ p_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{ R_w \}} \]

diversification distribution
Example: management with benchmark

\[ w \leftarrow w - b \]

weights \quad benchmark \quad weights

\[ \text{relative weights} \]

\[ \tilde{R} \equiv E^{-1} R \]
return of principal portfolios

\[ \tilde{w} \equiv E^{-1} w \]
weights of original portfolio on principal portfolios

\[ \begin{align*}
\nu_n & \equiv \tilde{w}_n^2 \lambda_n^2 \\
\sigma_n & \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{Sd \{R_w\}} \\
p_n & \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{R_w\}}
\end{align*} \]
variance concentration curve

\[ \uparrow \quad \downarrow \]
volatility / \textbf{tracking error} concentration curve

\[ \uparrow \quad \downarrow \]
diversification distribution
Example: management with benchmark

\[ w \rightarrow w - b \]

relative weights

\[ \tilde{R} \equiv E^{-1}R \]

return of principal portfolios

\[ \tilde{w} \equiv E^{-1}w, \]

weights of original portfolio on principal portfolios

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\begin{align*}
v_n & \equiv \tilde{w}_n^2 \lambda_n^2 \\
s_n & \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Sd} \{ R_w \}} \\
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variance concentration curve

\[ \updownarrow \]

volatility / tracking error concentration curve

\[ \updownarrow \]

diversification distribution
Example: management with benchmark

\[ w \rightarrow w - b \]

relative weights

\[ \tilde{R} \equiv E^{-1}R \]

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\]

variance concentration curve

\[ \leq \]

volatility / tracking error concentration curve

\[ \leq \]

diversification distribution
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MEAN-DIVERSIFICATION FRONTIER

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REFERENCES
A. MEUCCI - Managing Diversification  Mean-Diversification Frontier

\[ \tilde{R} \equiv E^{-1}R \]
return of principal portfolios

\[ \tilde{\mathbf{w}} \equiv E^{-1}\mathbf{w}, \]
weights of original portfolio on principal portfolios

\[ v_n \equiv \tilde{w}_n^2 \lambda_n^2, \]
variance concentration curve

\[ s_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Sd}\{R_w\}}, \]
volatility concentration curve

\[ p_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var}\{R_w\}}, \]
diversification distribution: “probability mass”
A. MEUCCI - Managing Diversification  Mean-Diversification Frontier

\[ \tilde{R} \equiv E^{-1}R \quad \text{return of principal portfolios} \]

\[ \tilde{w} \equiv E^{-1}w, \quad \text{weights of original portfolio on principal portfolios} \]

\[ v_n \equiv \tilde{w}_n^2 \lambda_n^2 \quad \text{variance concentration curve} \]

\[ s_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{std} \{R_w\}} \quad \text{volatility concentration curve} \]

\[ p_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{var} \{R_w\}} \quad \text{diversification distribution: “probability mass”} \]

\[ R_w \equiv \tilde{w}'\tilde{R}. \]
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Mean-Diversification Frontier

return of principal portfolios

weights of original portfolio on principal portfolios

entropy

\[ - \sum_{n=1}^{N} p_n \ln p_n \]

diversification distribution: “probability mass”

\[ \tilde{R} \equiv E^{-1}R \]

\[ \tilde{w} \equiv E^{-1}w \]

variance concentration curve

\[ v_n \equiv \tilde{w}_n^2 \lambda_n^2 \]

volatility concentration curve

\[ s_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Sd} \{ R_w \}} \]

principal portfolio number

\[ p_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{ R_w \}} \]
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Effective number of bets

$$N_{\text{Ent}} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right)$$

diversification index

entropy

$$\sum_{n=1}^{N} p_n \ln p_n$$
Effective number of bets

\[ N_{\text{Ent}} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

full concentration \[ N_{\text{Ent}} \approx 1 \]

weights

diversification distribution

\[ p_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{ R_w \}} \]

diversification distribution: “probability mass”
Effective number of bets

\[ N_{Ent} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

- full concentration: \( N_{Ent} \approx 1 \)
- full diversification: \( N_{Ent} \approx N \)

weights

diversification distribution

\[ p_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{ R_w \}} \]

diversification distribution: “probability mass”
Effective number of bets

\[ N_{\text{Ent}} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

Mean-diversification frontier

\[ w_\varphi \equiv \arg \max_{w \in C} \{ \varphi \mu' w + (1 - \varphi) N_{\text{Ent}}(w) \} \]
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Mean-Diversification Frontier

Effective number of bets

\[ N_{Ent} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

- full concentration \( N_{Ent} \approx 1 \)
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Mean-diversification frontier

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Allocation in terms of

original portfolio weights

not principal portfolios
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**Mean-Diversification Frontier**

Effective number of bets

\[ N_{Ent} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

- Full concentration: \( N_{Ent} \approx 1 \)
- Full diversification: \( N_{Ent} \approx N \)

**Transaction costs**

\[ \mu'w \rightarrow \mu'w - T(w, w_{cur}) \]

Non linear, non-continuous function of current and target portfolio

**Mean-diversification frontier**

\[ w_{\varphi} \equiv \arg\max_{w \in C} \{ \varphi \mu'w + (1 - \varphi) N_{Ent}(w) \} \]
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Mean-Diversification Frontier

**Effective number of bets**

\[ N_{\text{Ent}} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

- full concentration  \[ N_{\text{Ent}} \approx 1 \]
- full diversification  \[ N_{\text{Ent}} \approx N \]

**Transaction costs adjusted mean-diversification frontier**

\[ w_\varphi \equiv \arg\max_{w \in \mathcal{C}} \left\{ \varphi (\mu' w - T (w, w_{\text{cur}})) + (1 - \varphi) N_{\text{Ent}} (w) \right\} \]
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Mean-Diversification Frontier

**Effective number of bets**

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- Full diversification: \( N_{\text{Ent}} \approx N \)

**Transaction costs adjusted mean-diversification frontier**

\[ w_\varphi \equiv \arg\max_{w \in \mathcal{C}} \left\{ \varphi (\mu' w - T (w, w_{\text{cur}})) + (1 - \varphi) N_{\text{Ent}} (w) \right\} \]

**Effective number of bets**

**Expected return**
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Conditional Analysis

Constraints

\[ A \Delta w \equiv 0 \]

\[ K \times N \quad N \times 1 \]
Constraints

\[ A \Delta w \equiv 0 \]

\[ K \times N \quad N \times 1 \]

Conditional PCA

Feasible trades

\[ n = K + 1, \ldots, N \]

\[ e_n \equiv \text{argmax} \{ e' \Sigma e \} \]

\[ e'e = 1 \]

\[ e' \Sigma e_j = 0, \quad \text{for all existing } e_j \]

such that

\[ A e \equiv 0 \]
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Conditional Analysis

Constraints

\[ A \Delta w \equiv 0 \]

\[ K \times N \quad N \times 1 \]

Conditional PCA

Feasible trades

\[ n = K + 1, \ldots, N \]

\[ e_n \equiv \underset{e', e \equiv 1}{\text{argmax}} \{ e' \Sigma e \} \]

such that

\[ e' \Sigma e_j \equiv 0 \]

for all existing \( e_j \)

\[ A e \equiv 0 \]

Complementary, unfeasible trades

\[ n = 1, \ldots, K \]

\[ e_n \equiv \underset{e', e \equiv 1}{\text{argmax}} \{ e' \Sigma e \} \]

such that

\[ e' \Sigma e_j \equiv 0 \]

for all existing \( e_j \)
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- Article:
  Attilio Meucci, “Managing Diversification”
  *Risk* - May 2009

- MATLAB examples:
  MATLAB Central Files Exchange (see above article)

- This presentation:
  [www.symmys.com](http://www.symmys.com) > Teaching > Talks