BNP PARIBAS

Risk tolerance and optimal portfolio choice

Marek Musiela, BNP Paribas, London

Corporate
and Investment
Banking

Joint work with T. Zariphopoulou (UT Austin)

- **Investments and forward utilities, Preprint 2006**
- **Backward and forward dynamic utilities and their associated pricing systems: Case study of the binomial model, Indifference pricing, PUP (2005)**
- **Investment and valuation under backward and forward dynamic utilities in a stochastic factor model, to appear in Dilip Madan's Festschrift (2006)**

BNP PARIBAS

Contents

Corporate and Investment Banking

Investment banking and martingale theory

- **Ideal relationship**
- **Mathematical logic of the derivative business perfectly in line with the theory**
- **Pricing by replication comes down to calculation of an expectation with respect to a martingale measure**
- **Issues of the measure choice and model specification and implementation dealt with by the appropriate reserves policy**
- **However, the modern investment banking is not about hedging (the essence of pricing by replication)**
- **Indeed, it is much more about return on capital - the business of hedging offers the lowest return**

Investment banking and utility theory

- Dysfunctional relationship
- Mathematical utility theory formulated in a very abstract way and focused on solving problems of limited practical importance
- Economic utility theory formulated and developed in the context which is not directly focused on applications in investment banking
- When reformulated in the investment context it faces the difficulty to explain the intuitive meaning of utility
- Only very sporadic examples where utility was used in a pricing context
- To the best of my knowledge, extremely limited use in the asset allocation context

BNP PARIBAS

Main weaknesses

- No clear idea how to specify the utility function
- The classical or recursive utility is defined in isolation to the investment opportunities given to an agent
- Explicit solutions to the optimal investment problems can only be derived under very restrictive model and utility assumptions - dependence on the Markovian assumption and HJB equations
- The general non Markovian models concentrate on the mathematical questions of existence of optimal allocations and on the dual representation of utility
- No easy way to develop practical intuition for the asset allocation

Corporate and Investment Banking

Dynamic utility

- **U(x,t) is an adapted process**
- **As a function of x, U is increasing and concave**
- **For each self-financing strategy the associated (discounted) wealth satisfies**

$$E_P\left(U\left(X_t^{\pi},t\right)\middle|F_s\right)\leq U\left(X_s^{\pi},s\right) \qquad 0\leq s\leq t$$

- **There exists a self-financing strategy for which the associated (discounted) wealth satisfies**

$$E_P\left(U\left(X_t^{\pi^*},t\right)\middle|F_s\right)=U\left(X_s^{\pi^*},s\right) \qquad 0\leq s\leq t$$

Corporate
and Investment
Banking

Value function and dynamic utility

- **Value function**

$$V(x,t) = \sup_{\pi} E_P\left(u\left(X_T^{\pi}, T\right) \middle| F_t, X_t^{\pi} = x\right) \qquad 0 \le t \le T$$

- **Dynamic programming principle**

$$V\left(X_s^{\pi^*}, s\right) = E_P\left(V\left(X_t^{\pi^*}, t\right) \middle| F_s\right) \qquad 0 \le s \le t \le T$$

- **Dynamic utility coincides with the value function**

$$U(x,t) = V(x,t) \qquad x \in R, \qquad 0 \le t \le T$$

Corporate
and Investment
Banking

Difficulties

- Dynamic utility U(x,t) is defined by specifying the utility function u(x,T) and then calculating the value function
- The utility at time 0, i.e., U(x,0), may be very complicated and quite unintuitive.
- It depends strongly on the specification of the market dynamics
- The analysis of such utilities depends strongly on the Markovian assumption for the asset dynamics and the use of HJB equations
- Only very specific cases of such utilities, like exponential, can be analysed in a model independent way

BNP PARIBAS

Alternative approach – an example

- Start by defining the utility function at time 0, i.e., set $U(x,0)=u(x,0)$
- Define an adaptive process $U(x,t)$ by combining the variational and the market related inputs to satisfy the properties of a dynamic utility
- Benefits
  - The function $u(x,0)$ represents the utility for today and not for, say, ten years ahead
  - The variational inputs are the same for the general classes of market dynamics – no Markovian assumption required
  - The market inputs have direct intuitive interpretation
  - The family of such utilities is sufficiently rich to allow for thinking about allocations in a way which is model and utility independent

Corporate
and Investment
Banking

BNP PARIBAS

Variational inputs

- **Utility equation**

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

- **Risk tolerance equation**

$$r_t + \frac{1}{2} r^2 r_{xx} = 0, \qquad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

Corporate
and Investment
Banking

Market inputs

- **Investment universe of one riskless and k risky securities**
- **General Ito type dynamics for the risky securities**
- **Standard d-dimensional Brownian motion driving the dynamics of the traded assets**
- **Traded assets dynamics**

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right), \qquad i = 1,\ldots,k$$

$$dB_t = r_t B_t dt$$

Corporate
and Investment
Banking

Market inputs

- **Using matrix and vector notation assume existence of the market price for risk process which satisfies**

$$\mu_t - r_t 1 = \sigma_t^T \lambda_t$$

- **Benchmark process**

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t), \qquad Y_0 = 1, \qquad \sigma_t \sigma_t^+ \delta_t = \delta_t$$

- **Views (constraints) process**

$$dZ_t = Z_t \phi_t \cdot dW_t, \qquad Z_0 = 1$$

- **Subordination process**

$$dA_t = \left| \sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t \right|^2 dt, \qquad A_0 = 0$$

Corporate
and Investment
Banking

Alternative approach – an example

- **Under the above assumptions the process U(x,t), defined below is a dynamic utility**

$$U(x,t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

- **It turns out that for a given self-financing strategy generating wealth X one can write**

$$dU_t = dU(X_t, t) = \left(u_x \frac{Z}{Y}\sigma\pi - u_x \frac{XZ}{Y}\delta - U\phi\right)_t \cdot dW_t$$

$$+\frac{1}{2}u_{xx}Z\left|\frac{1}{Y}\sigma\pi - \left(\left(\frac{X}{Y} - R\right)\delta + R\sigma\sigma^{+}(\lambda+\phi)\right)\right|_t^2 dt$$

$$R_t = r\left(\frac{X_t}{Y_t}, A_t\right)$$

Corporate and Investment Banking

Optimal portfolio

- The optimal portfolio is given by

$$\frac{1}{Y_t}\pi_t^* = \sigma_t^+\left(\left(\frac{X_t^*}{Y_t} - R_t^*\right)\delta_t + R_t^*(\lambda_t + \phi_t)\right)$$

$$R_t^* = r\left(\frac{X_t^*}{Y_t}, A_t\right)$$

$$r_t + \frac{1}{2}r^2 r_{xx} = 0, \qquad r(x,0) = r_0(x)$$

- Observe that
  - The optimal wealth, the associated risk tolerance and the optimal allocations are benchmarked
  - The optimal portfolio incorporates the investor views or constraints on top of the market equilibrium
  - The optimal portfolio depends on the investor risk tolerance at time 0.

Portfolio dynamics

- **Assume that the following processes are continuous vector-valued semimartingales**

$$\sigma_t^+ \delta_t, \qquad \sigma_t^+ \lambda_t, \qquad \sigma_t^+ \phi_t, \qquad t \geq 0$$

- **Then, the optimal portfolio turns out to be a continuous vector-valued semimartingale as well. Indeed,**

$$\frac{1}{Y_t} \pi_t^* = \left(\frac{X_t^*}{Y_t} - R_t^* \right) \sigma_t^+ \delta_t + R_t^* \left(\sigma_t^+ \lambda_t + \sigma_t^+ \phi_t \right)$$

BNP PARIBAS

Corporate and Investment Banking

Wealth and risk tolerance dynamics

- The dynamics of the (benchmarked) optimal wealth and risk tolerance are given by

$$d\left(\frac{X_t^*}{Y_t}\right) = R_t^*\left(\sigma_t\sigma_t^+(\lambda_t+\phi_t)-\delta_t\right)\cdot\left((\lambda_t-\delta_t)dt+dW_t\right)$$

$$dR_t^* = r_x\left(\frac{X_t^*}{Y_t}, A_t\right)d\left(\frac{X_t^*}{Y_t}\right)$$

- Observe that zero risk tolerance translates to following the benchmark and generating pure beta exposure.

- In what follows we assume that the function r(x,t) is strictly positive for all x and t

Corporate
and Investment
Banking

Canonical variables

- **The wealth and risk tolerance dynamics can be written as follows**

$$d\left(\frac{X_t^*}{Y_t}\right) = R_t^* dM_t, \qquad dR_t^* = r_x\left(\frac{X_t^*}{Y_t}, A_t\right) dR_t^* dM_t$$

$$dM_t = \left(\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t\right) \cdot \left((\lambda_t - \delta_t) dt + dW_t\right)$$

- **Observe that**

$$d\langle M \rangle_t = dA_t$$

- **Introduce the processes**

$$x_1(t) = \frac{X^*_{A_t^{(-1)}}}{Y_{A_t^{(-1)}}}, \qquad x_2(t) = R^*_{A_t^{(-1)}}, \qquad w(t) = M_{A_t^{(-1)}}$$

Canonical dynamics

- **The previous system of equations becomes**

$$dx_1(t) = x_2(t)dw(t), \qquad x_1(0) = z \qquad z = \frac{x}{y}$$

$$dx_2(t) = r_x(x_1(t), t)x_2(t)dw(t), \qquad x_2(0) = r_0(z)$$

- **It turns out that it can be solved analytically**

BNP PARIBAS

Corporate
and Investment
Banking

Linear equation

- **Let h(z,t) be the inverse function of**

$$z \to \int_{z_0}^{z} \frac{1}{r(u,t)} du, \qquad h(\bullet,t) = \left(\int_{z_0}^{\bullet} \frac{1}{r(u,t)} du \right)^{(-1)}$$

- **It turns out that h(z,t) solves the following linear equation**

$$h_t + \frac{1}{2} h_{zz} - \frac{1}{2} r_x(z_0, t) h_z = 0$$

$$h(\bullet, 0) = \left(\int_{z_0}^{\bullet} \frac{1}{r_0(u)} du \right)^{(-1)}$$

Corporate and Investment Banking

Explicit representation

- **Solution to the system of equations is given by**

$$x_1(t) = h(z(t),t)$$
$$x_2(t) = h_z(z(t),t)$$
$$z(t) = h_0^{(-1)}(z) - \frac{1}{2}\int_0^t r_x(z_0, s)\,ds + w(t)$$

- **One can easily revert to the original coordinates and obtain the explicit expressions for**

$$\frac{X_t^*}{Y_t}, \qquad R_t^*$$

BNP PARIBAS

Corporate and Investment Banking

BNP PARIBAS

Optimal wealth

- **The optimal (benchmarked) wealth can be written as follows**

$$\frac{X_t^*}{Y_t} = x_1(A_t) = h(z(A_t), A_t) \qquad h(\bullet, t) = \left(\int_{z_0}^{\bullet} \frac{1}{r(u,t)} du \right)^{(-1)}$$

$$z(A_t) = h_0^{(-1)}(z) - \frac{1}{2} \int_0^t r_x(z_0, A_s) dA_s + M_t$$

$$dM_t = \left(\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t \right) \cdot \left((\lambda_t - \delta_t) dt + dW_t \right)$$

$$dA_t = d\langle M \rangle_t = \left| \sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t \right|^2 dt$$

Corporate and Investment Banking

BNP PARIBAS

Risk tolerance

- **The risk tolerance process can be written as follows**

$$R_t^* = x_2(A_t) = h_z(z(A_t), A_t) \qquad h(\bullet, t) = \left(\int_{z_0}^{\bullet} \frac{1}{r(u,t)} du\right)^{(-1)}$$

$$z(A_t) = h_0^{(-1)}(z) - \frac{1}{2}\int_0^t r_x(z_0, A_s) dA_s + M_t$$

$$dM_t = \left(\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t\right) \cdot \left(\left(\lambda_t - \delta_t\right) dt + dW_t\right)$$

$$dA_t = d\langle M \rangle_t = \left|\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t\right|^2 dt$$

Corporate and Investment Banking

Beta and alpha

- For an arbitrary risk tolerance the investor will generate pure beta by formulating the appropriate views on top of market equilibrium, indeed,

$$\sigma_t \sigma_t^{+} \left(\lambda_t + \phi_t \right) - \delta_t = 0 \quad \Rightarrow \quad d\left(\frac{X_t^{*}}{Y_t} \right) = 0, \quad dR_t^{*} = 0$$

- To generate some alpha on top of the beta the investor needs to tolerate some risk but may also formulate views on top of market equilibrium

No benchmark and no views

- The optimal allocations, given below, are expressed in the discounted with the riskless asset amounts

$$\pi_t^* = R_t^* \sigma_t^+ \lambda_t, \qquad R_t^* = r\left(X_t^*, A_t\right)$$

$$dA_t = \left|\sigma_t \sigma_t^+ \lambda_t\right|^2 dt$$

$$r_t + \frac{1}{2} r^2 r_{xx} = 0, \qquad r(x,0) = r_0(x)$$

- They depend on the market price of risk, asset volatilities and the investor's risk tolerance at time 0.

- Observe no direct dependence on the utility function, and the link between the distribution of the optimal (discounted) wealth in the future and the implicit to it current risk tolerance of the investor

Corporate and Investment Banking

No benchmark and hedging constraint

- The derivatives business can be seen from the investment perspective as an activity for which it is optimal to hold a portfolio which earns riskless rate

- By formulating views against market equilibrium, one takes a risk neutral position and allocates zero wealth to the risky investment. Indeed,

$$\delta_t = 0, \qquad \phi_t = -\lambda_t \qquad \Rightarrow \qquad \pi_t^* = 0$$

- Other constraints can also be incorporated by the appropriate specification of the benchmark and of the vector of views

No riskless allocation

- Take a vector such that

$$1 \cdot \sigma_t^+ \nu_t \neq 0$$

- Define

$$\phi_t = \frac{1 - 1 \cdot \sigma_t^+ \lambda_t}{1 \cdot \sigma_t^+ \nu_t} \nu_t, \qquad \delta_t = \sigma_t \sigma_t^+ (\lambda_t + \phi_t)$$

- The optimal allocation is given by

$$\pi_t^* = X_t^* \sigma_t^+ (\lambda_t + \phi_t)$$

- It puts zero wealth into the riskless asset. Indeed,

$$1 \cdot \pi_t^* = X_t^* 1 \cdot \sigma_t^+ \left(\lambda_t + \frac{1 - 1 \cdot \sigma_t^+ \lambda_t}{1 \cdot \sigma_t^+ \nu_t} \nu_t \right) = X_t^*$$

Steps to follow

- Specify the investment universe and its equilibrium dynamics
- Determine the current risk tolerance of an investor relatively to that universe (could try to imply it from the specification of future wealth distribution)
- Specify a benchmark and views or constraints
- Solve the FDE to recover the function r(x,t)
- Determine the variational input u(x,t) of the utility function, namely, solve

$$u_t u_{xx} = \frac{1}{2} u_x^2, \qquad u(x,0) = \int_0^x \exp\left(-\int_0^y \frac{1}{r(z,0)} dz \right) dy$$

- We set the utility of zero wealth at time zero to be zero and the slope of the utility at time zero for zero wealth to be equal to one. Of course other choices are possible

Steps to follow

- **Specify the dynamic forward utility by combining the variational input with the choice of a benchmark, views or constraints**
- **The optimal portfolio is optimal with respect to this utility**
- **Recover the function h(x,t), which is the inverse of the function**

$$x \to \int_0^x \frac{1}{r(u,t)} du$$

- **Specify the optimal wealth and risk tolerance processes**
- **Analyse the outcome and potentially recalibrate**

Corporate
and Investment
Banking

Disclaimer

- **The views expressed in this presentation are those of the author and not necessarily those of BNP Paribas**

Corporate
and Investment
Banking