Hybrid and Commodity Derivatives
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Introduction

- We will focus on pricing and risk management of exotic hybrid products based on futures and OTC swaps on Crude Oil (WTI, Brent) and metals (precious and industrial)

- For a general introduction to Commodity markets and models, see:

  - Helyette Geman (2005) *Commodities and Commodity Derivatives*. Wiley.

  - Vincent Kaminski (2004) *Managing Energy Price Risk*. Risk Books.

Multi-Asset Trading

Multi-Asset Trading

- Structured products on combinations of two or more asset classes
- Asset classes: Fixed Income, Commodities, Foreign Exchange, etc
- One of the largest derivative market growth area
- Retail distribution is driving the main demand (high net worth clients)

Himalaya Option

- For each period, return is based on the best performance of a basket of underlying assets
- At each period, the most performing asset drops out of the basket
- Local/global caps and floors can apply

Himalaya Option

- With global floor on total performance, we are long correlation (similar to a basket option)
- Without caps/floors, Himalaya options are a sum of Best-Of options, and therefore we are short correlation
- In the general case, correlation sensitivity can switch from positive to negative near a fixing date, even with small changes in underlying prices

Himalaya Option

- Gamma and cross-gamma can be significant
- Risk near fixing date is similar to that of barrier options
- Practical hedging: use Principal component risk rather than asset delta risk

Rainbow Option

- Payoff based on a weighted average of Commodity Index performances

- Weights depend on performance rank

Ex: 50% Best Index + 30% Asset 2 + 20% Asset 3

Auto-call Note

- N underlyings
- Note pays 15% per annum if all underylings are above a pre-fixed Strike, say 60%
- Every year before last, if all underlyings are above a pre-fixed Callable Level, say 100%, note redeems at 100%
- The last year, if all underlyings are above the Strike, note redeems at 100%, otherwise at worst performing underlying return + 100%

Main Sources of Risk

- First order greeks
- Gamma and cross-gamma
- Inter-asset correlations
- Quanto (fx-asset correlations)
- Skew / smile
- Parameter estimation: historical vs. risk neutral
- Liquidity / Transaction costs

Hybrid Derivatives Modeling

Hybrid Derivatives Modeling

One-fit-all type of models are desirable but highly unpractical:

- parameterization is constrained by calibration to market

- implementation of numerical methods is highly dependent on the specific model used (dynamics, number of parameters)

- In practice: a few generic classes of models (dynamics and pricing frameworks) seem sufficient

Ornstein-Uhlenbeck Models

- Multi-variate lognormal models are often used by practioners in multi-asset modeling:
- Motivation 1: closed form solutions are available - but they can be misleading: e.g. barrier option closed form formulas usually assume a constant drift in the dynamics of the underlying: not very realistic for many commodities!
- Motivation 2: it is convenient to use a multi-variate Ornstein-Uhlenbeck process in order to have correlated interest rates dynamics, or model multi-factor commodity curves.

Ornstein-Uhlenbeck Models

- Why use 2 or more factors?

  Front futures contract tend to trade at a higher volatility than back contracts and there is decorrelation along the future curve.

- Example 1: Gibson & Schwartz (1990):

$$\begin{cases} dX_t = (r_t - \delta_t - \frac{1}{2}\sigma_1^2)dt + \sigma_1 dW_t^1 \\ d\delta_t = K(\alpha - \delta_t)dt + \sigma_2 dW_t^2 \\ E(dW_t^1 dW_t^2) = \rho dt \end{cases}$$

Ornstein-Uhlenbeck Models

- Example 2: Casassus and Collin-Dufresne (2005)

$$\begin{cases} dr_t = (\mu_t - K_r r_t) + \sigma_r dW_t^r \\ d\delta_t = (K_{\delta 0}(t) + K_{\delta r} r_t + K_\delta \delta_t + K_{\delta X} X_t)dt + \sigma_\delta dW_t^\delta \\ dX_t = (r_t - \delta_t - \tfrac{1}{2}\sigma_X^2)dt + \sigma_X dW_t^X \end{cases}$$

Ornstein-Uhlenbeck Models

A very useful class of models are Ornstein-Uhlenbeck processes:

$$dX_t = (AX_t + b_t)dt + \sigma_t dW_t \qquad (1)$$

where X is a n-dimensional stochastic vector, W(t) is a vector of n Brownian motions, A a n x n mean-reversion matrix, b(t) a n-dimensional vector and σ(t) a n-dimensional volatility vector.

In addition Brownian motions are correlated:

$$E_t[dW_t dW_t^T] = \rho dt$$

Ornstein-Uhlenbeck Models

By Cholesky decomposition:

$$\sigma_t \rho \sigma_t^T = \Sigma_t \Sigma_t^T$$

and (1) can be rewritten as:

$$dX_t = (AX_t + b_t)dt + \Sigma_t d\tilde{W}_t$$

where

$$E_t[d\tilde{W}_t d\tilde{W}_t^T] = Idt$$

The solution of the SDE (1) is closed form, yielding closed form Monte Carlo sample paths:

$$X_T = D(t,T)X_t + \int_t^T D(u,T)b_u du + \int_t^T D(u,T)\sigma_u dW_u$$

Ornstein-Uhlenbeck Models

Under mild technical conditions, one can find a tensor φ and a vector α so that:

$$D_{ij}(t,T) = [e^{-A(T-t)}]_{ij} = \sum_k \sum_l \varphi_{ijkl} e^{\alpha_k (T-t)} (T-t)^l$$

[Cf. Yann Coatanlem, *Closed form solutions for Ornstein-Uhlenbeck processes*, Salomon Brothers, 1997]

In addition, closed form solutions are also available for the integral of the process:

$$\int_t^T X_u du = \Gamma(t,T) X_t + \int_t^T \Gamma(u,T) b_u du + \int_t^T \Gamma(u,T) \sigma_u dW_u$$

Zero coupon bond can also be computed analytically and don't need to be approximated in Monte-Carlo. When pricing American deals, lattices don't need to be constructed out to the end of the life of a swap or a bond, contrary to models like Black-Karasinski. That is a significant advantage.

Stochastic Volatility Models

Why using a stochastic volatility model?

- Unspanned stochastic volatility: Cf. Collin-Dufresne and Goldstein (2000) *Do bonds span the Fixed Income markets?*
- Stationary shape of the forward skew

Stochastic Volatility Models

- The following dynamics generalize most of the stochastic volatility models commonly used (beyond Affine…):

$$d\sigma_t^{\alpha} = k(\theta - \sigma_t^{\beta})dt + \nu\sigma_t^{\gamma} dW_t^{\sigma}$$

- For Monte-Carlo simulation, a second order Mistein scheme is preferred to the usual first order scheme:

$$\sigma_T^{\alpha} = \sigma_t^{\alpha} + k(\theta - \sigma_t^{\beta})\Delta t + \nu\sigma_t^{\gamma}\Delta W + \tfrac{1}{2}\nu\sigma_t^{\gamma}(\nu\sigma_t^{\gamma})^{T}.(\Delta W^{2} - \Delta t)$$

- In some special cases, the volatility SDE can be solved closed-form or semi-closed form: this will increase accuracy and enforce positivity constraints

Stochastic Volatility Models

- Typical Monte-Carlo time steps for a model with stochastic volatility will be one month or less
- Path-dependent / Bermudan payoffs: use the Longstaff-Schwartz method
- Quanto adjustment is the same as in deterministic vol case; under the domestic risk neutral measure:

$$\begin{cases} \dfrac{dS_t^f}{S_t^f} = (r_t^f - \lambda(t)\sigma_t^f)dt + \sigma_t^f d\widetilde{W}_t^f \\ \lambda(t) = \rho^{f,X}\sigma_X(t) \end{cases}$$

Beyond Gibson-Schwartz and Heston

- Helyette Geman (2000):

$$\begin{cases} dS_t = k(L_t - \ln S_t)S_t dt + \sqrt{V_t}\, dW_t^1 \\ \dfrac{dL_t}{L_t} = \mu_t dt + \sigma_2 dW_t^2 \\ dV_t = b(c - V_t)dt + \sigma_3 \sqrt{V_t}\, dW_t^3 \end{cases}$$

This is an extension of the Eydeland and Geman model (1998): L(t) was a fixed level: not so clear for oil in a bull cycle…

Beyond Gibson-Schwartz and Heston

- Richter and Sorensen (2002):

$$\begin{cases} dP_t = P_t[(r_t - \delta_t)dt + e^{\gamma(t)}\sqrt{v_t}\,dW_t^1] \\ d\delta_t = (\alpha(t) - \beta\delta_t)dt + e^{\gamma(t)}\sigma_\delta\sqrt{v_t}\,dW_t^2 \\ dv_t = (\theta - \kappa v_t)dt + \sigma_v\sqrt{v_t}\,dW_t^3 \end{cases}$$

with $$\alpha(t) = \alpha_0 + \sum_{k=1}^{K_\alpha}(\alpha_k \cos(2\pi kt) + \alpha_k^* \sin(2\pi kt))$$

and $$\gamma(t) = \sum_{k=1}^{K_\gamma}(\gamma_k \cos(2\pi kt) + \gamma_k^* \sin(2\pi kt))$$

Monte-Carlo Greeks

- At the very least, re-use same sample when bumping parameters…
- **Pathwise differentiation**

  - switching derivative and expectation operators

  - doesn’t work for non continuous payoffs

Monte-Carlo Greeks

- **Likelihood Ratio Method**

  - Broady / Glassermann, 1996

  - works for both non-continuous payoffs and path-dependent options

  - not easy to implement in the context of generic parsers…

$$\frac{\partial P}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int \pi(S(\alpha)) \psi(S) dS$$

$$= \int \pi(S) \frac{\partial \psi(S,\alpha)}{\partial \alpha} dS$$

$$= \int \left[\pi(S) \psi^{-1} \frac{\partial \psi}{\partial \alpha} \right] \psi(S,\alpha) dS$$

Model Calibration

Model Calibration

- Cross-sectional calibration: least Square Optimization Problem

$$\tilde{\theta} = \arg\min_{\theta} \sum_{i=1}^{N} \omega_i \left(P_i^{\mathrm{mod}el}(\theta) - P_i^{Market} \right)^2$$

- Use straddle opposed to calls or puts
- Weights should be a function of Bid-Ask spread:

good proxy = vega x volatility points

- Local Minima (1): use Simulated Annealing

Model Calibration

- Local Minima (2): transform a least square problem into a "more convex" problem in θ (set of parameters) in order to:

1. get closer to uniqueness
2. ensure stability of solution over time

- Example of quasi-convex problem: Baysian Fit:

$$\tilde{\theta} = \arg\min_{\theta} \sum_{i=1}^{N} \omega_i \left(P_i^{\text{model}}(\theta) - P_i^{Market} \right)^2 + \lambda \sum_{j=1}^{M} \left(\frac{\theta_j - \bar{\theta}_j}{\sigma_j} \right)^2$$

- How to determine trade-off between accuracy of calibration (MRSE) and stability ("entropy" term)?

Model Calibration

- Optimal trade-off: Morozov discrepancy principle (Cf. Rama Cont and Peter Tankov, *Financial Modelling with Jump Processes*, Chapman, 2004):

$$\lambda^* = Sup\left\{ \lambda : \varphi = \sum_{i=1}^{N} \omega_i \left(P_i^{\mathrm{model}}(\theta, \lambda) - P_i^{Market} \right)^2 \le \sum_{i=1}^{N} \omega_i \left(P_i^{Bid} - P_i^{Ask} \right)^2 \right\}$$

In practice φ is an increasing function of λ so the solution can be found by a simple search function – like Newton

Model Calibration

- Are we fitting too many parameters?
- Use Singular Value Decomposition
- If rank(Jacobian) < card(θ) => “co-linearity” in parameters:

  keep some parameters constant, i.e. estimated by Maximum Likelihood

Model Calibration

Estimating Correlations

- What correlations?
- Is there a term structure of correlation?
- Correlation regimes
- Should we use a stochastic correlation?
- Historical vs. risk neutral parameters
- Extreme events: common jumps across asset classes?

Model Calibration

Estimating Skew when no market data available…

Joint maximum likelihood of risk-neutral and physical parameters and risk-premia: Cf. Ait-Sahalia and Kimmel (2005) *Maximum likelihood estimation of stochastic volatility models*:

- application to general diffusion-based stochastic volatility models
- closed form expansions of transition functions
- approximated volatility state variables using short term implied ATM vols
- closed-form solution likelihood function (and asymptotic standard errors)

Model Calibration

Example: Risk neutral and physical parameters in the Heston model:

- Under the risk neutral measure, the dynamics are

$$d\begin{bmatrix} s_t \\ Y_t \end{bmatrix} = \begin{bmatrix} r_t - d_t - \frac{1}{2}Y_t \\ \kappa'(\gamma' - Y_t) \end{bmatrix} dt + \begin{bmatrix} \sqrt{(1-\rho^2)Y_t} & \rho\sqrt{Y_t} \\ 0 & \sigma\sqrt{Y_t} \end{bmatrix} d\begin{bmatrix} W_1^Q(t) \\ W_2^Q(t) \end{bmatrix}$$

- Under the physical measure

$$d\begin{bmatrix} s_t \\ Y_t \end{bmatrix} = \begin{bmatrix} a_t + bY_t \\ \kappa(\gamma - Y_t) \end{bmatrix} dt + \begin{bmatrix} \sqrt{(1-\rho^2)Y_t} & \rho\sqrt{Y_t} \\ 0 & \sigma\sqrt{Y_t} \end{bmatrix} d\begin{bmatrix} W_1^P(t) \\ W_2^P(t) \end{bmatrix}$$

- Assuming an affine risk premium

$$\Lambda = [\lambda_1\sqrt{(1-\rho^2)Y_t}, \lambda_2\sqrt{Y_t}]'$$

We have $a_t = r_t - d_t, \quad b = \lambda_1(1-\rho^2) + \lambda_2\rho - \frac{1}{2}, \quad \kappa = \kappa' - \lambda_2\sigma, \quad \gamma = \left(\frac{\kappa + \lambda_2\sigma}{\kappa}\right)\gamma'$

Hedging In Incomplete Markets

Options are not redundant...

- Options are not redundant in incomplete markets: there are risks that cannot be hedged away, even in continuous time trading

Merton approach

- Merton approach: *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics, 1976:

$$S_t = S_0 \exp[\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i]$$

Self-financing strategy is given by:

$$\phi_t = \frac{\partial C}{\partial S}(t, S_{t-})$$

Only the risk corresponding to the diffusion part is hedged…

Super-hedging

- If a contingent claim H cannot be replicated, a conservative approach is to construct a self-financing strategy ϕ which ends up almost surely with positive P&L:

$$P(V_T(\phi) = V_0 + \int_0^T \phi dS \geq H) = 1 \quad a.s.$$

- It can be shown (Kramkov, *Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets*, Prob. Theor. Relat. Fields, 1996) that the cost of super-hedging corresponds to the value of the option under the least favorable martingale measure

Super-hedging

- At a deal level, hedging can be expensive
- However, since the risk is NOT linear, super-hedging a portfolio of aggregate positions may be significantly cheaper than super-hedging each individual position

Quadratic Hedging

- Quadratic/Mean-variance Hedging:

$$\inf_{\phi} E[|V_T(\phi) - H|^2]$$

Examples:

- Discrete rehedging: Bouchaud/Potters/Sestovic, *Hedged Monte-Carlo: low variance derivative pricing with objective probabilities*, 2000

- Transaction costs: Leland, *Option pricing and replication with transaction costs*, Journal of Finance, 1985

- Delayed information: Chybiryakov and Gaussel, *How to hedge with a delayed information?,* 2003

In practice...

- Practical approach: hybrid approach between super-hedging and quadratic-hedging:

- Simulate a self-financing strategy and hedge using a quadratic approach in Monte-Carlo: imperfect trading approach $\Leftrightarrow$ control variate

- Use the distribution of the residual P&L given by the simulation and adjust the price by a pre-defined quantile of the distribution

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