Derivatives Replication Under Greenian Motion

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Disclaimer

- The material is this presentation is preliminary and the conclusions reached are tentative.

- The results obtained in no way reflect the opinions of employees of either Bloomberg or NYU.

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Overview

- There are five parts to this talk:
  1. Introduction
  2. Model-free Results on Calendar Spreads
  3. Derivatives Replication in Markovian Models
  4. Derivatives Replication under Greenian Motion
  5. Simplifications Arising under Laplace Dynamics

Part I - Introduction

- As is very well known, in models which go by the names of Cox Ross Rubinstein (CRR) and Black Merton Scholes (BMS), payoffs to derivative securities can be replicated by restricting price processes.
- For example, in discrete time, the CRR model assumes Bernoulli dynamics. In continuous time, the BMS model assumes continuous price paths.
- When the derivative security is written on the price path of a single underlying asset, model-based replication requires positions in just two assets.
- Parameter inputs are obtained either from time series (historical vol) or the market price of a single option (implied vol).

Markovian Models with Temporal and Spatial Homogeneity

- The CRR and BMS model are both Markovian Models in which the log price has stationary independent increments.

- When the term structure of ATM implied vol is not flat, temporal homogeneity can be relaxed and time-dependent CRR and BMS models can be used to price and hedge derivative securities.

- Now the model is calibrated to a whole term structure of option prices, but when the derivative security is written on the price path of a single underlying asset, model-based replication still requires positions in just two assets.

Strike Structure of Implied Vol

- When one or more implied volatility smiles are not flat, time-dependent CRR and BMS models are contradicted; a different kind of model is required.

- One of the more popular successors to time-dependent CRR and BMS models is the set of local vol models. These models further relax spatial homogeneity, but the basic nature of the motion is unaltered. In discrete time, the Derman Kani local vol model still assumes Bernoulli dynamics. In continuous time, the Dupire local vol model still assume no jumps. The Markovian nature of the dynamics is preserved.

- Local vol models are calibrated to a whole term and strike structure of option prices. When the derivative security is written on the price path of a single underlying asset, model-based replication still requires positions in just two assets.

What About Jumps?

- In binomial models, single period OTM options struck outside the allowed price range are worthless. In diffusion models, the analogous statement is that for every $\varepsilon > 0$:

$$\lim_{h \downarrow 0} \frac{\mathbb{Q}\{|S_{t+h} - S| > \varepsilon | S_t = S\}}{h} = 0,$$

  for all S in the domain of the diffusion. In words, as time to maturity goes to zero the prices of all OTM binary options decline towards zero sublinearly.

- Do near-dated OTM binary option prices behave this way in the data? In “A Simple Robust Test for the Presence of Jumps in Option Prices”, Liuren Wu and I find that for S&P500 options prices, the decline is linear, not sublinear, consistent with jumps being priced into near-dated OTM options. In fact, here’s a quote from former RFS editor Ken Singleton as he rejected our paper:

  “Every study that went looking for jumps has found them.”

Do Jumps Matter for Long Dated Options?

- Some people argue that jumps are required to price short-dated options, but that the effects of jumps on pricing wash out over long horizons. This argument mixes up jumps with independent increments, an assumption that is often made to make jumps tractable.

- Every long-dated option eventually becomes short-dated, but in markets with zero net supply, short-dated options need not be held.

- So do jumps necessarily matter for the pricing of long dated options?

An Illuminating Example

- Consider the problem of pricing a down-and-out call $DOC(K,M;L)$ with strike K, maturity M, and lower barrier $L \leq K$, written on a co-terminal forward price $F(M)$.

- In 1994, Bowie and Carr showed that when the underlying forward price cannot jump over the barrier, a down-and-out call struck at L has no time value, i.e.:

$$DOC_0(L,M;L) = B_0(M)[F_0(M) - L],$$

  where $B_0(M)$ is the discount factor.

- In contrast, when the underlying forward price can jump over the barrier, a down-and-out call has positive time value i.e.:

$$DOC_0(L,M;L) > B_0(M)[F_0(M) - L].$$

- Hence, any continuous model will underprice this down-and-out call, assuming only that it reprices the forward. The continuity assumption kills the convexity value of the DOC.

Hedging Under Jumps

- Assuming that the log price is affine in a standard Poisson process, Cox and Ross (1976) showed that the payoff of a derivative security written on the price path of a single underlying asset can still be replicated by dynamic trading in just two assets.

- If two jump sizes are possible at any instant, then *in general*, replication requires dynamic trading in three assets, eg. stock, bond, and variance swap.

- However, every rule has an exception, (except for this rule of course). For example, to replicate the payoff of a variance swap, one can show that one only needs a static position in the log contract and dynamic trading in the underlying.

- When markets are incomplete, there still exist payoffs which can be spanned in the existing structure eg. Put Call Parity. The discovery of the nature of such payoffs and their hedge is a subject of much practitioner interest.

Hedging Under a Continuum of Possible Jump Sizes

- The more jump sizes that are allowed, the more assets one needs to hedge.
- In this paper, we will focus on the case where a continuum of jump sizes is possible at any instant.
- Not surprisingly, for a derivative security written on the price path of a single underlying asset, perfect replication of the payoff will in general require dynamic trading in European options of all strikes and maturities up to that of the security.

Replicating Barrier Options

- Throughout this talk, we will focus on the problem of how to replicate the payoff to a barrier option via dynamic trading in European options.

- When the underlying asset has price dynamics that are Markovian in itself and time, we show that there exists a replicating strategy even though jumps of any size are possible at every time.

- In general, the replicating strategy has positions in European options of all strikes and maturities up to that of the barrier option. The positions in European options are altered as the options become near-dated. Otherwise, the holdings are static.

Greenian Motion

- When we add a particular structure to the jump dynamics called Greenian motion, we no longer require positions in intermediate maturity options struck away from the barrier. A memoryless property allows their role to be captured by options struck on the barrier.

- Furthermore, the model can be explicitly calibrated to a single observation of the market prices of European options of all strikes and maturities up to that of the barrier option.

- Under Greenian motion, the determination of option value functions only requires numerically solving a sequence of linear second order ODE's.

- When we further require deterministic volatility, we get closed form solutions for barrier option values and hedge ratios.

The Setting

- Without loss of generality, we will focus on the problem of how to replicate the payoff to a down-and-out call (DOC) struck above a lower barrier.

- For simplicity, we will work in discrete time and assume that barrier monitoring occurs on the same frequency as price changes.

- For simplicity, we will assume that the underlying is a forward price and that interest rates are zero.

- For simplicity, we let the underlying forward price go negative. Hence, at each period, the support of possible forward prices at each future time is the whole real line.

Consequences of Frictionless Markets and No Arbitrage

- We assume frictionless markets and in particular that market prices are unaffected by the trading of an individual.
- We assume no arbitrage and hence the existence of a probability measure $\mathbb{Q}$ under which (forward) prices are martingales.
- Initially, we will not place further restrictions on the dynamics of the underlying forward price.

Butterfly Spreads

- Let $n = 0, 1, \ldots, \infty$ index the natural numbers and denote discrete calendar time.
- For $n = 0, 1, \ldots, \infty$, let F_n be the price at time n of some underlying asset.
- For $n = 0, 1, \ldots, m$, suppose that one can trade calls paying $(F_m - K)^+$ at its maturity m. Let $C_n(F;K,m)$ denote the conditional market price at time n of a European call of strike $K \in \mathbb{R}$ and maturity $m \geq n$. The market price is conditional on $F_n = F$ and is in general random for $n > 0$. The conditional call price is also not directly observable for $n > 0$ or $F \neq F_0$, but can be observed at $n = 0$ and $F = F_0$.
- Suppose that for each date n and each maturity $m \geq i$, the call pricing function $C_n(F;K,m)$ is C^2 in K. Then Breeden and Litzenberger (1978) show that:

$$\mathbb{Q}\{F_m \in dK | F_n = F\} = \frac{\partial^2}{\partial K^2} C_n(F;K,m) dK.$$

Butterfly Spreads (Con'd)

- Recall that:
$$\mathbb{Q}\{F_m \in dK | F_n = F\} = \frac{\partial^2}{\partial K^2} C_n(F; K, m) dK.$$

- In words, the conditional risk-neutral density of the future forward price F_m at level K is given by the market price at time n of a butterfly spread paying $\delta(F_n - K)$ at its maturity $m \geq n$.

- Are there any other model-free results concerning option spreads?

Calendar Spread of Binary Calls

- Recall that a butterfly spread maturing at m pays $\delta(F_m - K)$ when n crosses m from below. Thus a cash inflow of infinite magnitude arises if $F_m = K$ at this time.

- If time could run backward, then this infinite inflow would turn into an outflow if $F_m = K$ as n crosses m from above.

- In contrast to time, a price F can cross a given level K from above or below.

- Reversing the roles of space and time, we now show that a calendar spread of binary calls has the same action as a butterfly spread of calls.

Parsifal*: I hardly tread- though it seems I already have come far.
Gurnemanz: You see, my son, here time becomes space.

From the first act of the opera Parsifal by Wolfgang Wagner.

*I thank Lane Hughston for suggesting this quote.

Calendar Spread of Binary Calls (Con'd)

- For $n = 0, 1, \ldots$, let $BC_n(F; K, m) \equiv -\frac{\partial}{\partial K} C_n(K, m)$ denote market prices at time n of binary calls of all strikes $K \in \mathbb{R}$ at each discrete maturity $m \geq n$ Each binary call pays $\mathcal{H}(F_m - K)$ at its maturity m, where $\mathcal{H}(x)$ is the Heaviside function.

- For $m \geq n+1$, consider the value at time n of a calendar spread of binary calls when the two maturities are adjacent: $BC_n(F; K, m+1) - BC_n(F; K, m)$.

- Examining the $2 \times 2 = 4$ cases, we find that the payoff at time $m+1$ is:

$$\mathcal{H}(K - F_m)\mathcal{H}(F_{m+1} - K) - \mathcal{H}(F_m - K)\mathcal{H}(K - F_{m+1}).$$

- In words, one dollar is received at time $m+1$ if F crosses K from below, but one dollar is paid at time $m+1$ if F crosses K from above.

Calendar Spread of Standard Calls

- The results on the last slide imply that $BC_n(F;K,m+1) - BC_n(F;K,m)$ is the conditional price at time n of purchasing a claim that provides both a unit increase in value at K if F upcrosses K just after time m and a unit decrease in value if F downcrosses K just after time m.

- On the next slide, we show that $C_n(F;K,m+1) - C_n(F;K,m)$ is the conditional price at time n of purchasing a claim that provides both a unit increase in slope if F upcrosses K just after time m and a unit decrease in slope if F downcrosses K just after time m.

Calendar Spread of Standard Calls (Con'd)

- For $m \geq n+1$, consider the value at time $n = 0, 1, \ldots$ of a calendar spread of calls when the two maturities are adjacent: $BC_n(F;K,m+1) - BC_n(F;K,m)$.
- Examining the $2 \times 2 = 4$ cases, we find that the payoff at time $m+1$ is:

$$\mathcal{H}(K - F_m)(F_{m+1} - K)^+ + \mathcal{H}(F_m - K)(K - F_{m+1})^+.$$

- In words, $F_{m+1} - K$ dollars are received at time $m+1$ if F crosses K from below, while $K - F_{m+1}$ dollars are received at time $m+1$ if F crosses K from above.
- As the slope ratcheting has the same sign as the price change, the payoffs are positive in both cases.

Down-and-Out Calls and Generalizations

- For $n = 0, 1, \ldots$, let $DOC_n(F; K, M; L)$ denote the conditional value at time n of a down-and-out call struck at $K \geq L$ and maturing at $M \geq n$. The down-and-out call pays $\mathcal{H}(\min_{n \in [0,M]} F_n - L)(F_M - K)^+$ at its maturity date M.

- More generally, for $n = 0, 1, \ldots$, let $DOC_n^p(F; K, M; L, m)$ denote the conditional value at time n of a partial barrier down-and-out call struck at $K \geq L$. Monitoring of the out barrier begins at a future date $m \geq n$ and ends at the maturity date $M \geq m$. The partial barrier down-and-out call pays $\mathcal{H}(\min_{n \in [m,M]} F_n - L)(F_M - K)^+$ at its maturity date M.

- The concept of a partial barrier down-and-out call (PBDOC) generalizes both a down-and-out call and a standard call since:

$$DOC_n^p(F; K, M; L, n) = DOC_n(F; K, M; L).$$

$$DOC_n^p(F; K, M; L, M) = C_n(F; K, M).$$

Calendar Spread of Partial Barrier Calls

- Recall that the PBDOC pays $\mathcal{H}(\min_{n\in[m,M]} F_n - L)(F_M - K)^+$ at its maturity M.

- Suppose that at time n, we calendar spread a PBDOC on its monitor start date. The conditional price at time n of this spread is $DOC_n^p(F;K,M;L,m+1) - DOC_n^p(F;K,M;L,m)$

- Examining the $2 \times 2 = 4$ possible realizations of the ordered pair $(\mathcal{H}(F_{m+1} - K), \mathcal{H}(F_m - K))$, we find that $DOC_n^p(F;K,M;L,m+1) - DOC_n^p(F;K,M;L,m)$ is the conditional price at time n of a claim that pays $\mathcal{H}(L - F_m)\mathcal{H}(F_{m+1} - L)DOC_{m+1}(F_{m+1};K,M;L)$ at time $m+1$.

- In words, the spread pays the value of a down-and-out call on an upcross of L just after m and the spread pays zero otherwise.

Replicating a Down-and-Out Call

- Recall that the partial barrier down-and-out call (PBDOC) pays $\mathcal{H}(\min_{n\in[m,M]} F_n - L)(F_M - K)^+$ at its maturity date M.

- It is a tautology that $DOC_0^p(F;K,M;L,M) = DOC_0^p(F;K,M;L,0)$

$$+\sum_{m=0}^{M-1}[DOC_0^p(F;K,M;L,m+1) - DOC_0^p(F;K,M;L,m)].$$

- Hence, $C_0(F;K,M) = DOC_0(F;K,L;M)$

$$+\sum_{m=0}^{M-1} E^{\mathbb{Q}}[\mathcal{H}(L-F_m)\mathcal{H}(F_{m+1}-L)DOC_{m+1}(F_{m+1};K,M;L)|F_0].$$

Replicating a Down-and-Out Call (Con'd)

- Recall that $C_0(F;K,M) = DOC_0(F;K,L;M)$
$$+\sum_{m=0}^{M-1} E^{\mathbb{Q}}[\mathcal{H}(L-F_m)\mathcal{H}(F_{m+1}-L)DOC_{m+1}(F_{m+1};K,M;L)|F_0].$$

- But by Taylor series with remainder, $\mathcal{H}(F_{m+1}-L)DOC_{m+1}(F_{m+1};K,M;L)$:
$$\begin{aligned} &= \frac{\partial}{\partial F}DOC_{m+1}(L;K,M;L)(F_{m+1}-L)^+ \\ &\quad + \int_L^{\infty} \frac{\partial^2}{\partial F^2}DOC_{m+1}(J;K,M;L)(F_{m+1}-J)^+ dJ. \end{aligned}$$

- Hence, $E^{\mathbb{Q}}\mathcal{H}(L-F_m)\mathcal{H}(F_{m+1}-L)DOC_{m+1}(F_{m+1};K,M;L)|F_0] =:$
$$E^{\mathbb{Q}}\left\{\mathcal{H}(L-F_m)\frac{\partial}{\partial F}DOC_{m+1}(L;K,M;L)[C_m(F_m,L,m+1)-C_m(F_m,L,m)]|F_0\right\}$$
$$+E^{\mathbb{Q}}\left\{\mathcal{H}(L-F_m)\int_L^{\infty}\frac{\partial^2}{\partial F^2}DOC_{m+1}(J;K,M;L)[C_m(F_m;J,m+1)-C_m(F_m;J,m)]dJ|F_0\right\}.$$

Replicating a Down-and-Out Call (Con'd)

- Substitution implies: $C_0(F;K,M) = DOC_0(F;K,L;M) + \sum_{m=0}^{M-1} E^{\mathbb{Q}}\{f_m(F_m)|F_0\}$, where

$$\begin{aligned} f_m(F) &\equiv \mathcal{H}(L-F)\frac{\partial}{\partial F}DOC_{m+1}(L;K,M;L)[C_m(F,L,m+1)-C_m(F,L,m)] \\ &+ +\mathcal{H}(L-F)\int_L^{\infty}\frac{\partial^2}{\partial F^2}DOC_{m+1}(J;K,M;L)[C_m(F;J,m+1)-C_m(F;J,m)]dJ. \end{aligned}$$

- We now assume that the underlying forward price is Markov in itself and time.
- Consequently, the value functions for C and DOC at any future time and price are determined by numerically solving a partial integro difference equation (PIDE).
- As the DOC's delta and gamma also become determined, each function f_m becomes known.

Replicating a Down-and-Out Call (Con'd)

- Recall that the equation at the top of the previous slide had the form

$$C_0(F;K,M) = DOC_0(F;K,L;M) + \sum_{m=0}^{M-1} E^{\mathbb{Q}}\{f_m(F_m)\},$$

where f is a known function of its argument.

- Using the Breeden and Litzenberger result, we can hold a portfolio of butterfly spreads of all strikes below the barrier and all maturities up to that of the DOC.

$$DOC_0(F_0;K,M;L) = C_0(F_0;K,M) - \sum_{m=1}^{M} \int_{-\infty}^{L} N^{bs}(F,m) \frac{\partial^2}{\partial K^2} C(F_0;F,m)dF,$$

where:

$$\begin{aligned} N^{bs}(F,m) &= \frac{\partial}{\partial F} DOC_{m+1}(L;K,M;L)[C_m(F,L,m+1) - C_m(F,L,m)] \\ &+ \int_L^{\infty} \frac{\partial^2}{\partial F^2} DOC_{m+1}(J;K,M;L)[C_m(F;J,m+1) - C_m(F;J,m)]dJ. \end{aligned}$$

Semi-Static Hedging

- To show how the hedge works, note that if the underlying never crosses the barrier before M, then all of the butterflies expire worthless while the standard call gives the desired payoff.

- If the underlying crosses the barrier before M, then at the first passage time to the barrier, the standard option portfolio is worthless. The reason is that one can use the payoff from each maturing butterfly spread to buy near-dated OTM calls struck at L and above. These call positions finance transitions into a short position in an alive down-and-out call held only in the continuation region. This self-financing semi-static trading strategy in barrier options has a final payoff equal to a short call payoff. The long call hedges this liability and thus the standard option portfolio must have zero value for stock prices below L at any time prior to T.

Drawbacks of the General Markovian Hege

- When hedging the sale of a DOC, an obvious drawback of the general Markovian approach is that it requires the ability to take positions in puts of all strikes below the barrier.

- A second drawback is that it is not clear how to choose the Markov model. In theory, one can calibrate to the prices of standard options at all strikes and maturities at all past times. However, the inversion problem is too big and the data is probably lacking.

- A third drawback is computational - one has to numerically solve a PIDE to implement the hedge.

Greenian Motion

- We introduce a new type of Markovian dynamics called “Greenian Motion”(GM) as a way to overcome all three drawbacks of general Markovian hedging.
- We will see that under GM, positions in all intermediate maturity options struck away from the barrier are no longer needed in the hedge.
- Furthermore, the risk-neutral dynamics of the underlying forward price process can be explicitly identified on day 0 from the initial market prices of standard options of all strikes and maturities up to that of the DOC.
- Finally, on the computational side, one need only numerically solve a sequence of second order linear ordinary differential equations (ODE’s) to obtain both values and hedge ratios.

Review of Green's Functions

- Let $\mathcal{A}_F$ be the following linear second order differential operator:
$$\mathcal{A}_F f(F) \equiv \frac{a^2(F)}{2} f''(F) - f(F).$$

- Let $g(F;K)$ be the Green's function of $\mathcal{A}_F$, i.e. $g(F;K)$ solves the 2nd order ODE:
$$\frac{a^2(F)}{2}\frac{\partial^2}{\partial F^2} g(F;K) - g(F;K) = -\delta(F-K),$$
subject to the boundary conditions: $\lim_{F\to\pm\infty} g(F;K) = 0$, for all $K \in \mathbb{R}$.

- For $n = 1, 2, \ldots$, we may define a sequence of Green's functions denoted $g(F,n,K)$ as solutions to the following sequence of linear second order ODE's:
$$\frac{a^2(F,n)}{2}\frac{\partial^2}{\partial F^2} g(F,n,K) - g(F,n,K) = -\delta(F-K),$$
subject to the boundary conditions: $\lim_{F\to\pm\infty} g(F,n,K) = 0$, for all $n = 1, 2, \ldots$ and $K \in \mathbb{R}$.

Adjoint Equation

- Recall that for each $n = 1, 2 \ldots$, the Green's function $g(F, n, K)$ solves the ODE:

$$\mathcal{A}_F g(F,n,K) \equiv \frac{a^2(F,n)}{2}\frac{\partial^2}{\partial F^2} g(F,n,K) - g(F,n,K) = -\delta(F-K),$$

subject to the boundary conditions: $\lim_{F \to \pm\infty} g(F,n,K) = 0.$

- By a well known theorem which deserves a name, when g is considered as a function of K, it solves the adjoint equation:

$$\mathcal{A}_F^* g(F,n,K) \equiv \frac{\partial^2}{\partial K^2}\left[\frac{a^2(K,n)}{2} g(F,n,K)\right] - g(F,n,K) = -\delta(F-K),$$

subject to the boundary conditions: $\lim_{K \to \pm\infty} g(F,n,K) = 0.$

- The solution $g(F,n,K)$ is nonnegative. On the next slide, we show that when considered as a function of K, g is a PDF.

Integral Transform

- Recall that $g(F,n,K)$ solves the adjoint equation:

$$\mathcal{A}_F^* g(F,n,K) \equiv \frac{\partial^2}{\partial K^2}\left[\frac{a^2(K,n)}{2} g(F,n,K)\right] - g(F,n,K) = -\delta(F-K),$$

subject to the boundary conditions: $\lim_{K\to\pm\infty} g(F,n,K) = 0.$

- For p real, define the integral transform $M(F,n,p) \equiv \int_{-\infty}^{\infty} e^{pK} g(F,n,K)dK$ as another nonnegative function. We implicitly restrict the function $a(K,n)$ so that the integral exists for p in a neighborhood of the origin.

- To obtain an equation governing M, suppose we multiply the top ODE by e^{pK} and integrate K over $\mathbb{R}$.

Integral Transform (Con'd)

- As a result, $M(F,n,p)$ solves the equation:

$$\int_{-\infty}^{\infty} e^{pK} \frac{\partial^2}{\partial K^2} \left[\frac{a^2(K,n)}{2} g(F,n,K) \right] dK - M(F,n,p) = -e^{pF}.$$

- Integrating by parts twice and assuming that:

$$\lim_{K \to \pm\infty} \frac{\partial}{\partial K} \left[\frac{a^2(K,n)}{2} g(F,n,K) \right] = 0, \qquad \lim_{K \to \pm\infty} \frac{a^2(K,n)}{2} g(F,n,K) = 0,$$

  we get that $M(F,n,p)$ solves the equation:

$$\frac{p^2}{2} \int_{-\infty}^{\infty} e^{pK} a^2(K,n) g(F,n,K) dK - M(F,n,p) = -e^{pF}.$$

- Evaluating at $p = 0$ implies that when considered as a function of K, g is a probability density function (PDF):

$$\int_{-\infty}^{\infty} g(F,n,K) dK = 1, \text{ for all } F \in \mathbb{R}, n = 1, 2 \ldots$$

First Moment

- Since g is a PDF in K, $M(F,n,p) \equiv \int_{-\infty}^{\infty} e^{pK} g(F,n,K)dK$ is the moment generating function (MGF) of the random variable whose PDF is g.

- Recall that the MGF $M(F,n,p)$ solves the equation:

$$\frac{p^2}{2}\int_{-\infty}^{\infty} e^{pK} a^2(K,n) g(F,n,K)dK - M(F,n,p) = -e^{pF}.$$

- Differentiating once w.r.t. p and setting $p=0$ implies that:

$$\int_{-\infty}^{\infty} Kg(F,n,K)dK = F,$$

  for all $F \in \mathbb{R}$ and $n = 1,2\ldots$.

- Thus, the first moment of the mystery random variable is F for all n.

Second Moment and Variance

- Recall again that the MGF $M(F,n,p)$ solves the equation:

$$\frac{p^2}{2}\int_{-\infty}^{\infty} e^{pK}a^2(K,n)g(F,n,K)dK - M(F,n,p) = -e^{pF}.$$

- Differentiating twice w.r.t. p and setting $p=0$ implies that:

$$\int_{-\infty}^{\infty} K^2 g(F,n,K)dK = F^2 + \int_{-\infty}^{\infty} a^2(K,n)g(F,n,K)dK,$$

  for all $F \in \mathbb{R}$ and $n = 1,2\ldots$

- As a result, we have that:

$$\int_{-\infty}^{\infty} (K-F)^2 g(F,n,K)dK = \int_{-\infty}^{\infty} a^2(K,n)g(F,n,K)dK,$$

  for all $F \in \mathbb{R}$ and $n = 1,2\ldots$

What is Greenian Motion?

- Consider a discrete time Markov process F defined over a finite time horizon.
- We say that F follows *Greenian Motion* if for every n, its transition PDF at time n is the Green's function $g(F,n,K)$, i.e.:

$$\mathbb{Q}\{F_{n+1} \in dK | F_n = F\} = g(F,n,K).$$

- Since $\int_{-\infty}^{\infty} K g(F,n,K) dK = F$, we see that F is a martingale:

$$E^{\mathbb{Q}}[F_{n+1} | F_n = F] = F.$$

Interpreting a^2

- Suppose we interpret F as a continuous time process which only jumps at integer times.

- When a^2 is constant, the distribution of each increment is Laplace with mean zero and variance a^2.

- When a^2 is constant, the process has independent increments and enjoys a scaling property, viz $c(F_n - F_0)$ has the same law as $F_{c^2 n} - F_0$. As a consequence, we can alternatively draw from a Laplace with unit standard deviation and assign the outcome to a time step of length $1/a^2$. European options should have the same price.

- When a^2 depends on F and n, we may continue to draw from a standard Laplace and have the result apply to random time steps of length $1/a^2(F_n, n)$.

Link to Time Homogeneous Diffusion Martingale

- Recall that a discrete time Markov martingale F follows Greenian Motion if for every n, its transition PDF solves the inhomogeneous ODE:

$$\mathcal{A}_F g(F,n,K) \equiv \frac{a^2(F,n)}{2}\frac{\partial^2}{\partial F^2} g(F,n,K) - g(F,n,K) = -\delta(F-K),$$

subject to the boundary conditions: $\lim_{F\to\pm\infty} g(F,n,K) = 0.$

- By Feynman Kac, we may for every fixed $n = 1,2,$ express g as the following conditional expectation:

$$g(F,n,K) = E^{\mathbb{Q}}\left[\int_0^\infty e^{-u}\delta(M_u^{(n)} - K)du | M_0^{(n)} = F\right],$$

where $M^{(n)}$ is a time homogeneous diffusion process solving the stochastic differential equation (SDE):

$$dM_u^{(n)} = a(M_u^{(n)}, n)dW_u^{(n)}, \qquad u \geq 0,$$

where $\{W_u^{(n)}, u \geq 0\}$ is a $\mathbb{Q}$ standard Brownian motion.

Link to a Time Inhomogeneous Diffusion Martingale

- Instead of linking the n-th increment in F with the entire trajectory of a time homogeneous diffusion $M^{(n)}$, we may link the whole F process with a time inhomogeneous diffusion D, which is locally time homogeneous.

- Suppose that D is a time inhomogeneous diffusion whose diffusion coefficient jumps from $a(D,n)$ to $a(D,n+1)$ when an independent and completely standard Poisson process N jumps from level n to level $n+1$.

- By running the diffusion process D on N, we obtain the discrete time process F.

- This link makes clear why F is a local martingale and it also permits a familiar interpretation of the function $a(F,n)$ as a diffusion coefficient.

Identifying the Process from Options Prices

- Recall that a discrete time Markov process F follows Greenian Motion if for every n, its transition PDF solves the inhomogeneous ODE:

$$\mathcal{A}_K^* g(F,n,K) \equiv \frac{\partial^2}{\partial K^2}\left[\frac{a^2(K,n)}{2} g(F,n,K)\right] - g(F,n,K) = -\delta(F-K),$$

subject to the boundary conditions: $\lim_{K\to\pm\infty} g(F,n,K) = 0$.

- Integrating twice in K and recalling our assumption that:

$$\lim_{K\to\pm\infty} \frac{\partial}{\partial K}\left[\frac{a^2(K,n)}{2} g(F,n,K)\right] = 0, \qquad \lim_{K\to\pm\infty} \frac{a^2(K,n)}{2} g(F,n,K) = 0,$$

lifts the outer derivatives, while not introducing boundary terms. But from the definition of Greenian motion:

$$g(F,n,K) = \mathbb{Q}\{F_{n+1} \in dK | F_n = F\} = \frac{\partial^2}{\partial K^2} C_n(F;K,n+1).$$

Identifying the Process (Con'd)

- Substituting in the Breeden and Litzenberger result implies that:

$$\frac{a^2(K,n)}{2}\frac{\partial^2}{\partial K^2}C_n(F;K,n+1) - C_n(F;K,n+1) = -(F-K)^+.$$

- Taking risk-neutral expected value at time 0, we obtain the following forward partial differential difference equation (PDDE) for call prices:

$$\frac{a^2(K,n)}{2}\frac{\partial^2}{\partial K^2}C_0(F;K,n+1) - C_0(F;K,n+1) = -C_0(F;K,n).$$

- Hence the function $a^2(K,n)$ can be explicitly determined from call prices:

$$a^2(K,n) = \frac{2[C_0(F;K,n+1) - C_0(F;K,n)]}{\frac{\partial^2}{\partial K^2}C_0(F;K,n+1)}.$$

- Notice in contrast to Derman and Kani that the state space is continuous, and in contrast to Dupire, that the required data is discrete in maturity.

Backward PDDE for Butterfly Spread Values

- Recall once again that a discrete time Markov martingale F follows Greenian Motion if for every n, its transition PDF solves the inhomogeneous ODE:
$$\mathcal{A}_F g(F,n,K) \equiv \frac{a^2(F,n)}{2}\frac{\partial^2}{\partial F^2}g(F,n,K) - g(F,n,K) = -\delta(F-K),$$
subject to the boundary conditions: $\lim_{F\to\pm\infty} g(F,n,K) = 0.$

- Let $BS_n(F;K,M) \equiv \mathbb{Q}\{F_M \in dK | F_n = F\}$ be the conditional value at time $n = 0,1,\ldots$ of a butterfly spread paying $\delta(F_M - K)$ at its maturity $M \geq n$.

- Multiplying the top ODE by $BS_{n+1}(K;G,M)$ and integrating K over $\mathbb{R}$ implies that long dated butterfly spread values solve the backward PDDE:
$$\frac{a^2(F,n)}{2}\frac{\partial^2}{\partial F^2}BS_n(F;G,M) - BS_n(F;G,M) = -BS_{n+1}(F;G,M),$$
subject to $\lim_{F\to\pm\infty} BS_n(F;G,M) = 0$ and $BS_M(F;G,M) = \delta(F-G)$.

Backward PDDE for Claim Values

- Recall that butterfly spread values solve the backward PDDE:

$$\frac{a^2(F,n)}{2}\frac{\partial^2}{\partial F^2}BS_n(F;G,M) - BS_n(F;G,M) = -BS_{n+1}(F;G,M),$$

subject to $\lim_{F\to\pm\infty} BS_n(F;G,M) = 0$ and $BS_M(F;G,M) = \delta(F-G)$.

- The conditional value at time n of a European-style claim paying $f(F_M)$ at its fixed maturity date M is:

$$V_n^f(F;M) = \int_{-\infty}^{\infty} f(G)\mathbb{Q}\{F_M \in dG | F_n = F\} = \int_{-\infty}^{\infty} f(G) BS_n(F;G,M)dG.$$

- Hence, multiplying the ODE by $f(G)$ and integrating G over the real line implies that conditional claim values satisfy the same backward PDDE:

$$\frac{a^2(F,n)}{2}\frac{\partial^2}{\partial F^2}V_n^f(F;M) - V_n^f(F;M) = -V_{n+1}^f(F;M),$$

subject to $V_M^f(F;M) = f(F)$.

Backward PDDE for Barrier and Bermudan Option Values

- We may similarly numerically solve a sequence of ODE's for discretely monitored knockout barrier options. In the knockout region, the continuation value is set to zero.
- Bermudan options are handled similarly; before recursing backward, the continuation value is replaced by the maximum of it and the reward for exercising early.

Summary

- Working in discrete time and under zero rates, we first presented some new model-free results concerning the values of calendar spreads of binary, standard, and partial barrier calls.
- We then showed how one can replicate the payoff to a down-and-out call when the underlying price process is Markov in itself.
- Since we allowed the possibility of jumps of all sizes, the replicating portfolio required positions in options of all strikes and maturities.
- By restricting dynamics down to Greenian motion, the calibration, valuation, and replication problems all simplify quite dramatically.
- An open question concerns the impact that the Greenian Motion assumption has on the range of permitted dynamics.

Other Properties of Greenian Motion

- Although we presented GM in discrete time, it has a continuous time counterpart being worked our presently. This continuous time counterpart is needed when barriers are monitored continuously.

- Although we presented GM with just one underlying source of uncertainty, one can also generalize it to multiple sources of uncertainty (eg. stochastic vol or multiple underlyings). Unlike its Markovian generalization, the single factor GM approach "leaves room" for hedging an additional uncertainty. Now one solves a sequence of elliptic PDE's rather than ODE's.

- Finally, the single factor GM approach is semi-analytic: using variation of parameters, one can write down formulas for values and hedge ratios in terms of a solution to a linear second order homogeneous ODE with variable coefficients. If the ODE has a closed form solution, then so do claim values and hedge ratios.