Cash Flow CLOs without Monte Carlo Simulation
Open Source XVA/Risk
How to Avoid Monte-Carlo and What to Do when You Can’t

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Columbia University
IEOR Financial Engineering Practitioners Seminar

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Some Introductory Thoughts

Part I: How to price CLO’s without Monte Carlo Simulation
- What is a CLO
- Why (Semi) Analytic results are important
- How does the Q-Q approximation work

Part II: XVA and Open Source Risk
- What is XVA, how does it relate to Exposure and Capital
- How do we use Monte Carlo Solve to this problem
- Why an Open Source Risk Library is a good thing

This talk is about options in unlikely places
Acknowledgements

Collaborators

- Dr. Chris Kenyon, Lloyds Bank
- Prof. James Gleeson, University of Limerick
- Dr. Roland Lichters, Quaternion
- Dr. Roland Stamm, Quaternion
- Rory Villiers, Quaternion

All views expressed are personal.
We absolutely agree with Prof. Derman’s view on Financial Models

- Financial Models are not **Theories** (in the sense of e.g. General Relativity) they are at best **Analogies**
- We have an obligation to understand and test their limitations and assumptions
- We have an obligation to be clear about what we’re sweeping under the rug

Some observations:

- "No models" are worse that "imperfect models" (as long as you understand the imperfection)
- We should strive to do "better" as practitioners
- Warning: This talk will contain some of the ’Baddies’ of financial crisis
- We will talk about correlation and Copula’s (e.g. Gaussian) but these **not central** to the results
What is a CLO?

- Structured Tranched Investment
- Investment Secured on a large Pool of Loans (held in trust)
- Investors buy Securities whose cashflows come directly from underlying Loans
- Securities have rank order from Senior, through Mezzanine to Equity.
- Senior Investors have first rights to interest and principal repayments.
- Structural Features (IC and OC) offer additional protection to Senior Investors
- We will be more precise in a few slides.
Why Are CLO’s Important?

U.S. CLO Issuance: Annual

as of 01/05/2016

$ B

130
120
110
100
90
80
70
60
50
40
30
20
10
0


Source: Intex, S&P, Moody’s, Wells Fargo Securities, LLC

BSL CLO MM CLO

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Why do we care about fast (semi) analytical pricing?

1. To understand the instrument and its risks
2. To compute risk sensitivities quickly
3. To include the instrument in Monte Carlo XVA and Capital calculations
4. Because we can (i.e. for fun)
To fix ideas let’s think of a Fixed Rate Loan.

- \( t \) or \( t_0 \) is today
- \( t_i \) is usually reserved for the payment dates of a trade (e.g. loan)
- \( P(t, t_i) \) is the discount factor at time \( t_i \) as seen today, \( t \)
- \( T \) or \( t_n \) is the last payment date or maturity of the loan.
- \( N \) is the Notional (Principal amount) of the loan
- \( K \) the fixed coupon (interest rate times notional) on the loan

Default Free NPV of the Loan

\[
\Pi(t) = \sum_{i=1}^{n} KP(t, t_i) + NP(t, t_n)
\]
What happens if the Loan issuer defaults?

- Call the time of default $\tau$
- All future payments on the loan cease
- Some time after $\tau$, the loan holder receives some fraction (Recovery) of the investment back from the liquidator
You can purchase protection against this unhappy outcome!

- It’s called a Credit Default Swap
- Investor pays a periodic fee, \( s\% \) (an insurance premium)
- If default occurs the investor receives \( (1 - R)N \)
- CDS fee implies the probability of default.

\[ s \implies Pr(\tau < t). \]
Joint Default

We have a pool (portfolio) of names:
Factors we should take into account are

▶ We know the default probabilities of the single names (marginal distributions)
▶ However the names may be more or less correlated
▶ In contagion (panic) scenarios all assets become correlated and fall at the same time.
What is a Synthetic CDO?

Simpler and more dangerous cousin of the CLO

Underlying Pool of Names

Funded or Unfunded CDO Tranches

| Argentina | 100m |
| Brazil    | 100m |
| Venezuela | 50m  |
| Gazprom   | 200m |
|           |      |
| Total     | 1bn  |

Equity Tranche
0-10% of Notional
7% +500 bp

Mezzanine Tranche
10-30% of Notional
100 bp

Senior Tranche
30-100% of Notional
40 bp
What is a Synthetic CDO?

Simpler and more dangerous cousin of the CLO

<table>
<thead>
<tr>
<th>Underlying Pool of Names</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina 100m</td>
</tr>
<tr>
<td>Brazil 100m</td>
</tr>
<tr>
<td>Venezuela 50m</td>
</tr>
<tr>
<td>Gazprom 200m</td>
</tr>
<tr>
<td>Total 1bn</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Funded or Unfunded CDO Tranches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity Tranche 0-10% of Notional</td>
</tr>
<tr>
<td>Mezzanine Tranche 10-30% of Notional</td>
</tr>
<tr>
<td>Senior Tranche 30-100% of Notional</td>
</tr>
</tbody>
</table>

| 7% +500 bp |
| 100 bp |
| 40 bp |
Q: What do we need to price a CDO?

A: The probability distribution (P) of Losses (L) on the portfolio.
Payoff from a CDO

Say we know the loss distribution (see later), how do we use it?

What’s that Payoff function?

- we start to lose Notional when $L$ hits $A$, the attachment point
- we continue to lose notional as $L$ increases
- when $L$ hits $D$ we are wiped out and can lose no further notional

What’s our expected Loss?
Expected Notional

Well our expected tranche loss is

\[
\mathbb{E}(\text{TRANCHE\_PAYOFF})(t) = \int_0^\infty \text{TRANCHE\_PAYOFF}(x)P(x, t)dx
\]

\[
= \int_A^D xP(x, t)dx + \int_D^\infty (D - A)P(x, t)dx
\]

with \(P(x, t)\) the loss probability density.
How to Compute the Loss Distribution

Warning: Models Behaving Badly

1. Specify a Joint Default Model
   ▶ Copula Models
     ▶ Gaussian Copula
     ▶ Gamma Copula
     ▶ your favourite other Copula
     ▶ Peng & Kou Model (2006)
     ▶ A. N. Other of your choosing

2. Compute the Loss Distribution
   ▶ Using Monte-Carlo Simulation (v. slow, many samples needed)
   ▶ Using (Semi) Analytic Methods (v. fast)
     ▶ e.g. Loss Bucketing (Hull White 2004)
     ▶ e.g. Transform (Laplace or Fourier) methods
     ▶ Only requirement is Conditional Independence of the joint default model

N.B. There exist very fast and accurate methods that don’t involve MC
and are not Gaussian Copula’s
What is a CLO? Part II

100M Columbia 5% 2020
75M Caltech 4.5% 2021
125M Harvard 6% 2025
50M MIT Libor +100 2030

Interest Account
Y
All Interest Flows

Principal Account
X
All principal and Recovery payments

Cashflow Waterfall

100M Equity Tranche
Excess Cash Quarterly

200M Mezz Tranche
Quarterly 5.5%

700M Senior Tranche
Quarterly 3.5%

1Bn Portfolio
Interest Waterfall:

1. Taxes
2. Trustee fees and expenses subject to cap
3. Administration fees and expenses subject to cap
4. Payments for hedge transactions other than early termination
5. Interest and fees under the liquidity facility
6. Senior servicing fee
7. Interest due on **Senior** notes
8. Redemption of **Senior** notes if over-collateralisation or interest coverage tests not met (sufficient to ensure tests are met)
9. Interest due on **next most Senior** notes
10. Redemption of **next most Senior** notes if over-collateralisation or interest coverage tests not met (sufficient to ensure tests are met)
11. :
12. Excess to the **Equity** notes.
Principal Waterfall:

1. Unpaid items in the first six items of the Interest Waterfall
2. Unpaid Interest on the Senior notes
3. Redemption of the Senior notes if OC and IC tests not met (sufficient to ensure tests are met)
4. Purchase of additional securities during the re-investment period
5. Redemption of the Senior notes
6. Redemption of the next most Senior notes

7. :
8. Unpaid Subordinated Portfolio servicing fees
9. Redemption of the Equity notes
What’s Different About a Cashflow CLO?

- Underlying portfolio consists of actual loans paying interest into the portfolio
- Two random variables Principal Redemptions ($X$) and Interest ($Y$)
- Structural Features such as IC and OC tests
- No automatic link between underlying portfolio defaults and tranche payoff

=> Most folks go straight to Monte Carlo Simulation. However
- MC is painfully slow (even with variance reduction) = tens of minutes
- How do you incorporate in risk and capital calculations that require MC (nested MC)
- It doesn’t help with understanding the structure of the problem

=> Hold off a little bit and think
Pricing Formula without Structural Features (IC, OC)

NPV depends on principal redemptions \(X\) and interest flows \(Y\) on the portfolio

\[
X := \sum_{j=1}^{m} Red^j(t).
\]

\[
Y := \sum_{j=1}^{m} C^j(t_i) 1_{\{\tau_j \geq t_i\}}.
\]

where

\[
Red^j(t) = R^j N^j 1_{\{\tau_j < t\}} 1_{\{t < T_j\}} + N^j \left[ R^j 1_{\{\tau_j < T_j\}} + 1_{\{\tau_j \geq T_j\}} \right] 1_{\{t \geq T_j\}}.
\]

We can easily and quickly get the univariate distributions of \(X\) and \(Y\) using semi-analytic methods.
The expected tranche coupon can be written in terms of $X$ and $Y$ as the two-dimensional integral

$$\mathbb{E}[C_k(t_i)] = \int_0^\infty \int_0^\infty g_k(x, y) f_{XY}(x, y) \, dx \, dy,$$

where $f_{XY}$ is the joint probability density function of $X$ and $Y$, and the tranche coupon function $g_k(x, y)$ is given, for $k = 1, \ldots, n - 1$ by:

$$g_k(x, y) = \min \left( K_k(t_i) \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - x, 0 \right), 0 \right), 0 \right),$$

$$\max \left( y - \sum_{l=1}^{k-1} K_l(t_i) \max \left( N_l(0) + \min \left( \sum_{j=1}^{l-1} N_j(0) - x, 0 \right), 0 \right), 0 \right).$$

This is a 2-D piecewise linear **European option pay-off.**
Figure: Available Interest at $t_2$ versus Pool Redemption at $t_1$ for a portfolio, and the Q-Q approximation curve. A recovery rate of $R_i = 0.01$ is assumed for all assets.
Known from the Statistics community as the Quantile-Quantile plot (Q-Q)

\[ x(t) = F_X^{-1}(t) \quad \text{and} \quad y(t) = F_Y^{-1}(1 - t), \quad \text{for } t \in [0, 1], \]

Given this approximation the 2-D integral reduces to

\[
\langle g(x, y) \rangle_{QQ} = \int_0^\infty g\left(F_X^{-1}(1 - F_Y(y)), y\right) f_Y(y)dy.
\]

=> Calculation time falls from tens of minutes to seconds or less
Some Results: Test Portfolio

The deal is a three tranche structure with a maturity of five years, quarterly fixed coupon payments and total notional 100 million units. The size, fixed coupon rates, interest coverage and overcollateralisation ratios are set out in table 1 below.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Notional</th>
<th>Coupon Rate</th>
<th>IC Ratio</th>
<th>OC Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (Senior)</td>
<td>80M</td>
<td>3% fixed</td>
<td>1.1</td>
<td>1.1</td>
</tr>
<tr>
<td>B (Mezzanine)</td>
<td>10M</td>
<td>8% fixed</td>
<td>1.05</td>
<td>1.05</td>
</tr>
<tr>
<td>C (Equity)</td>
<td>10M</td>
<td>Excess Interest</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table: Test Deal Characteristics

The underlying portfolio is a pool of 100 non-amortising bonds each of notional one million units with quarterly fixed rate payments of 3% and 5 year maturity. Each bond has a probability of default derived from a flat hazard rate curve and a deterministic recovery rate associated with it. These probabilities of default (PD) and recovery rates are varied in each of the scenarios set out in table 2 below. The yield curve used is a 2% flat forward curve.
The market data for this trade consists of default probabilities for each bond, recovery rates, correlation parameters and a yield curve. The base case and market scenarios that we apply are set out in table 2. In scenarios 8 and 9 we examine the impact of heterogeneous hazard rate and recovery rate, in these scenarios each bond is assigned a uniformly random rate in the range set out in the table.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Hazard Rate (bp)</th>
<th>R</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Base)</td>
<td>500</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>2 (Correlation up)</td>
<td>500</td>
<td>0.4</td>
<td>0.8</td>
</tr>
<tr>
<td>3 (Correlation down)</td>
<td>500</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>4 (Hazard rate up)</td>
<td>1000</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>5 (Hazard rate down)</td>
<td>250</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>6 (Recovery rate down)</td>
<td>500</td>
<td>0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>7 (Recovery rate up)</td>
<td>500</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>8 (Heterogeneous Hazard Rate)</td>
<td>250-1000</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>9 (Heterogeneous Recovery Rate)</td>
<td>500</td>
<td>0.2-0.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

**Table:** Market Data Scenarios
CS01 of Tranches

N.B. Using Gaussian Copula for Ease of Comparison

<table>
<thead>
<tr>
<th>Tranche</th>
<th>CS01 (bp) without IC/OC</th>
<th>CS01 (bp) with IC/OC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>A</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>C</td>
<td>14</td>
<td>14</td>
</tr>
</tbody>
</table>

Table: Credit Spread Sensitivity in Scenario 1
## Results Without Structural Features

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Q-Q</th>
<th>MC</th>
<th>Δ (bp)</th>
<th>Q-Q</th>
<th>MC</th>
<th>Δ</th>
<th>Q-Q</th>
<th>MC</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Scenario 1</td>
<td>Scenario 2</td>
<td>Scenario 3</td>
<td>Scenario 4</td>
<td>Scenario 5</td>
<td>Scenario 6</td>
<td>Scenario 7</td>
<td>Scenario 8</td>
<td>Scenario 9</td>
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<td>Portfolio</td>
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<td>99.81</td>
<td>-3</td>
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<td>-3</td>
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<td>103.01</td>
<td>102.98</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 3: Price Results and differences in basis points without IC and OC
## Results With Structural Features

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Q-Q</th>
<th>MC</th>
<th>Δ (bp)</th>
<th>Q-Q</th>
<th>MC</th>
<th>Δ</th>
<th>Q-Q</th>
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<tbody>
<tr>
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<td></td>
<td></td>
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<td><strong>Scenario 3</strong></td>
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<td>C</td>
<td>105.46</td>
<td>105.88</td>
<td>-42</td>
<td>82.84</td>
<td>82.93</td>
<td>-9</td>
<td>96.12</td>
<td>96.24</td>
<td>-12</td>
</tr>
</tbody>
</table>

Table 4: Price Results with IC and OC
What Are The Advantages

- **Understanding:** A CLO is just a (complex) European Option on $X$ and $Y$ that can be understood
- **Speed:** 23 Mins for 100,000 Sobol MC iterations vs 2 sec for same inherent error
- **Risk:** Can now comfortably do risk calculations
- **Capital:** Can include in an MC calculation now for Capital or XVA
Modern Derivatives Pricing and Credit Exposure Analysis
Theory and Practice of CSA and XVA Pricing, Exposure Simulation and Backtesting
Roland Lichters, Roland Stamm, Donal Gallagher

Hardcover  9781137494832  £60.00  /  $95.00

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Part II: Open Source XVA
There are several open source Pricing Libraries (e.g. QuantLib)
There is no shared open source risk library
Risk is the discipline that would benefit most from open source
No proprietary advantage in good Risk Models
Public benefit in transparency and consistency
Opportunity to share and peer review state of the art
Opposite of convergence: A plurality of models to choose from
Shared Infrastructure: No need to reinvent the wheel
Open to the research and academic communities
Coming Soon: Open Risk

- Quaternion will release an Open Source Risk Library: Open Risk
- Release due 2nd quarter 2016
- Designed for Large Scale Monte Carlo Simulation
- Cross Asset Class
- Covers Vanilla Products (Exotic extensions available)
- Applications: XVA, Basel III Exposure, Market Risk
- C++ Library
- XML Data Loading (Trade and Market Data)
- Excel Front End for ease of use
- Methodology book
These things are very closely related and involve European Options on Price Distributions

- CVA: The price adjustments for counterparty defaultability in a derivative
- XVA = CVA, DVA, FVA, MVA, KVA, TVA (Credit, Debit, Funding, Margin, Capital, Tax)
- Counterparty exposure and limit monitoring (PFE)
- Basel III Counterparty Risk Capital
- Market Risk

All have one thing in common: The option is too hard to price analytically
CVA as an example

CVA is the price adjustment for counterparty defaultability of a derivative

\[ NPV^D = NPV - CVA \]

Think about a single derivative with a counterparty that can default at a random time \( \tau \)

\[ CVA(t) = \mathbb{E}^Q \left[ 1_{t < \tau \leq T} \cdot (1 - R(\tau)) \cdot D(t, \tau) \cdot NPV^+(\tau) \mid \mathcal{G}_t \right] . \]

with a few simplifications

\[ CVA(t) = (1 - R) \sum_{i=1}^{n} (S(t, t_{i-1}) - S(t, t_i)) \mathbb{E}^Q \left[ D(t, t_i)NPV^+(t_i) \mid \mathcal{F}_t \right] \]

or a sum of European option prices!

\[ \mathbb{E}^Q \left[ D(t, t_i)NPV^+(t_i) \mid \mathcal{F}_t \right] . \]
So what’s so difficult?

So all I have to do is compute

\[ \mathbb{E}^Q \left[ D(t, t_i)NPV^+(t_i) \mid \mathcal{F}_t \right]. \]

How hard can this be? But.....

- the underlying is not a single trade but a netting set of many (thousands) trades of different types
- the derivative may be in many currencies => Multi Currency Model
- the derivatives may be across many asset classes => Multi Asset Class Models
- the derivatives may be collateralized i.e. \( NPV^+ \) reduced by cash posted
- the collateral may be complex (e.g. bonds)

=> Fiendishly complicated European Option
Single 10Y swap

10Y IR Swap - Expected Exposure

- Unsecured
- VM secured
- VM + IM held

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Typical Portfolio Distribution

NPV Distribution (Risk-Neutral LGM (Single Step)) - 20130102
What are the Ingredients I
What are the Ingredients II

- Cross-Asset Evolution Models
- Quaternion will start with industry standard SDE’s H-W, J-Y, CIR, Heston...
- Scenario Generation Framework
- Trade and Market Data Loading (XML based)
- Trade Pricing based on QuantLib
- Post Processing Applications to generate CVA, EE, PFE, DIM...
- A User Interface (Excel)
- and
- An enthusiastic community of Users and Contributors
- All users and contributors very welcome
Final Thoughts

- We can make progress on hard problems without using Monte-Carlo Simulation
- Using analytical methods improves understanding and risk insight
- When we have use Monte-Carlo there is a Public Good in a shared Open Risk Library
Thank you for your attention
Loss Distribution: Simplest Case

To fix ideas, let's think of a simple case:

- two assets with equal notional, 100M EUR
- Both with equal probability of default to time $T$, 1/2.
- Independent default
- Recovery Rate = 0.

What's the loss distribution?

<table>
<thead>
<tr>
<th>Loss</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$100M$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$200M$</td>
<td>$1/4$</td>
</tr>
</tbody>
</table>

In other words a binomial distribution:

What's the expected loss?

$$\mathbb{E}(L) = 0 \cdot \frac{1}{4} + 100 \cdot \frac{1}{2} + 200 \cdot \frac{1}{4} = 100M.$$
For $n$ assets all with equal probability of default $p$ and notional $N$, the loss distribution is the binomial distribution

$$Pr(r \text{ defaults}) = \binom{n}{r} p^r (1 - p)^{n-r}.$$ 

If the notional and probabilities of default are not equal then we get a combinatorial explosion of possible loss amounts but in theory each is known
So far we have been dealing with independent defaults.

- We know the marginal distributions
- Let’s form a joint distribution with a correlation parameter
- Common approach is a copula function.
- Most common copula is the Gaussian Copula

Mathematically

\[ P(\tau_1 < T, \tau_2 < T, \ldots) = \Phi_\rho(\Phi^{-1}(P_1(T)), \Phi^{-1}(P_2(T)), \ldots) \]

where

\[ \Phi_\rho \]

is the joint normal distribution with correlation \( \rho \)
**Copula Model** Let $P_j(t_j)$ be the cumulative probability that asset $j$ will default before time $t_j$. In a one-factor Copula model, consider random variables (asset values)

$$Y_j = \sqrt{\rho} M + \sqrt{1 - \rho} Z_j$$  \hspace{1cm} (1)

where $M$ and $Z_j$ are standard normal variables and $0 \leq \rho \leq 1$. The correlation between $Y_j$ and $Y_k$ is then $\rho$.

Let $\Phi_{Y_j}(y)$ be the cumulative normal distribution function of $Y_j$. $y$ is mapped to $t$ such that percentiles match, i.e. $\Phi_{Y_j}(y) = P_j(t)$ or $y = \Phi_{Y_j}^{-1}(P_j(t))$.

$$Prob\ (y_j < y | M) = \Phi \left( \frac{y - \sqrt{\rho} M}{\sqrt{1 - \rho}} \right)$$

or

$$Prob\ (t_j < t | M) = \Phi \left( \frac{\Phi_{Y_j}^{-1}(P_j(t)) - \sqrt{\rho} M}{\sqrt{1 - \rho}} \right)$$  \hspace{1cm} (2)
In order to drive correlation between the times to default, we will introduce a single systematic risk factor, $M$. We will calculate all probabilities conditional on $M$ and integrate with respect to $M$ to compute the global distribution. We note that, conditional on $M$, all assets are independent, and so we can determine $L(x, t|M)$. In other words

$$L(x, t) = \int_{-\infty}^{\infty} L(x, t|m)g(m)dm,$$

where $L(x, t|m)$ can be calculated on the assumption of Independent defaults.
So our problem is reduced to calculating the loss distribution conditional on $M$, where all the defaults are independent. There is a nice algorithm used to control the combinatorial explosion of possible outcomes and this is the loss bucketing algorithm (Hull & White 2003)...... see later.
Loss Bucketing Algorithm

```c++
for (Size i = 0; i < nominals.size(); i++) {
    Real L = nominals[i];
    Real P = probabilities[i];
    for (int k = a.size() - 1; k >= 0; k--) {
        if (p[k] > 0) {
            int u = locateTargetBucket (a[k] + L, k);
            Real dp = p[k] * P;
            if (u == k)
                a[k] += P * L;
            else {
                if (u < int(nBuckets_)) {
                    if (dp > 0.0) {
                        Real f = 1.0 / (1.0 + (p[u]/p[k]) / P);
                        a[u] = (1.0 - f) * a[u] + f * (a[k] + L);
                    }
                    p[u] += dp;
                }
                p[k] -= dp;
            }
        }
    }
}
```
Copula Integral

\[
\text{Distribution integral}(\text{const } \& f, \\
\text{const std::vector<Real>& nominals,} \\
\text{const std::vector<Real>& probabilities}) \text{ const} \{ \\
calculate();
\]

\[
\text{Distribution dist}(f.\text{buckets}(), 0.0, f.\text{maximum}()); \\
\text{for (Size i = 0; i < steps(); i++)} \{ \\
\text{std::vector<Real> conditional} \\
\quad = \text{conditionalProbability}(\text{probabilities}, m(i)); \\
\text{Distribution d = f(nominals, conditional);} \\
\text{for (Size j = 0; j < dist.size(); j++)} \\
\text{dist.addDensity}(j, d.\text{density}(j) \times \text{densitydm}(i)); \\
\} \\
\text{return dist;} \\
\}