“I am not interested in proofs, but only in what nature does” -Paul Dirac

Aim of the Course

The past six years of the Great Financial Crisis have led to a great interest in the failures and successes of financial models. People have blamed financial engineers for building bad models, for relying on them, for carrying out orders blindly, and so on. Some of this, but not all of it, is true.

Part of the aim of this course is to study the volatility smile. But another bigger aim is to understand how to use financial models, how to understand when you can rely on them and when you can’t.

This course uses mathematics, but it isn’t a course about mathematics, differential equations or stochastic calculus. The aim is to use mathematics to understand the world. I want to not merely study the solutions and methods of solution of a variety of options models, but also develop intuition about what they suppose and how to use them. No assumptions behind financial models are genuinely true, and no financial models are really correct, so it’s very important to understand what you’re doing and why you’re doing it.

This is a course about several themes:
1. The nature of financial modeling
2. Understanding volatility as a quality, a quantity, and an asset
3. Understanding the practical use of the Black-Scholes-Merton model. There’s more to it than just knowing the equation and its solution.
4. Understanding the successes and limitations of the Black-Scholes model.
5. Coming to grips with the volatility smile
6. The extensions of the Black-Scholes model to accommodate/explain the volatility smile.
7. Understanding the consequences of these extensions. It’s easy to make up new and richer models but we want to understand whether they are realistic, whether they are advantageous, and what they lead to.

I’d like you to come out of this course knowing how to use models and knowing when to use them and knowing how much you can rely on them, and having some sense of their efficacy and their fallibility. I’d also like you to know something about how to make new models.
Preamble

According to Prof. Steve Ross of MIT, one of the inventors of the binomial options pricing model, risk-neutral valuation and arbitrage pricing theory, “...options pricing is the most successful theory not only in finance, but in all of economics.” And it is indeed. But successful doesn’t mean it can make predictions like models in the natural sciences. People who think that end up in trouble, as we’ve seen in the past few years.

Academics in finance tend to think of options valuations as a solved problem, of little academic interest anymore. But for those of you who end up working as practitioners – on options trading desks in equities, fixed income, currencies or commodities, as risk monitors or risk managers or controllers or model auditors – you’ll find that it isn’t really a solved problem at all. Financial markets disrespect the Black-Scholes-Merton results even while they use its language. They disrespect all models. Most academics who haven’t lived or died by models don’t see this clearly, but practitioners and traders who are responsible for coming up with the prices for which they are willing to trade securities, especially exotic illiquid securities, grapple with this every day. They have to figure out how to amend the model’s results to cope with the real world.

In this class we’re going to talk a lot about the Black-Scholes model and its discontents. In one sense the Black-Scholes model is a total miracle: it lets you value, in a totally rational way, securities that before its existence had no clear value. In the Platonic world of Black-Scholes-Merton – a world with normal returns, geometric Brownian motion, infinite liquidity, continuous hedging and no transactions costs – it provides a method of synthesizing an option and it works perfectly. It’s a masterpiece of engineering in a world that doesn’t quite exist, because markets don’t obey all of its assumptions. But it is a model, not reality.

Some assumptions are violated approximately, and some more dramatically. The assumptions that you can hedge continuously, at zero transaction cost, are approximations we can adjust for, and we’ll illustrate that. Skilled traders and quants do this with a mix of estimation and intuition every day. For example, you can practically adjust for transactions costs by adding some dollars to your price, or some volatility points to the Black-Scholes formula. In that sense the model is pretty robust -- you can perturb it from its Platonic view of the world to take account, approximately, of the in-reality less modelable aspects of that world.

And the Black-Scholes model is so rational, given its assumptions, and has such a strong grip on everyone’s imagination that even people who don’t totally believe in it use it to quote prices they are willing to trade at. When you deal with models, quoting is always a problem and an opportunity. If you choose the right currency or numeraire to quote with, models can be simpler. For example, the US dollar is the standard currency for quoting gold prices, at least for the present, but if demand were mostly coming from Europe while the dollar is weakening, then the gold price might look constant in Euros and increasing in dollars. In that case the Euro might be a better base for building gold models. Similarly bond traders quote bond prices in terms of yield to maturity, the current constant forward rate at which you must discount a bond’s payoff to get its price. Once upon a time modeling the stochastic evolution of yields to maturity was the state of the art in modeling bond prices and interest rates; then people began to realize that this could lead to arbitrage violation and negative forward rates; nowadays, we have more sophisticated bond pricing models...
based off stochastic positive forward rates. Yet, yield to maturity is still a convenient metric for quoting bond prices, though not necessarily the foundation for a good model. In the same way, markets use the Black-Scholes formula to quote options prices, even though the model has its flaws.

But there are fundamental problems with the Black-Scholes-Merton model. For example, stocks don’t follow geometric Brownian motion. If they did, our problems would be over. They can jump, their distributions have fat tails … Some people even believe that a stock price’s variance is infinite rather than finite, in which case all bets are off. This is a big issue, one we won’t tackle much here. What we will focus on especially is the problem of the volatility smile.

Prior to the stock market crash of October 1987, the Black-Scholes implied volatility of equity options varied little with strike, though it did vary with expiration. Ever since that crash, the behavior of implied volatility in equity index markets has changed: market participants now think of implied volatility as a two-dimensional surface whose level at any time is a function of strike and expiration. This surface, with combined term and strike structure, is called the volatility smile, or sometimes the volatility skew. Here’s a vintage smile surface from fifteen years ago; I’ll show more recent ones in class.

**FIGURE 1.1.** The implied volatility surface for S&P 500 index options as a function of strike level and term to expiration on September 27, 1995.

This so-called volatility smile, initially a feature of equity index options markets only, has now become a feature of not just equity index options markets (CAC, DAX, Nikkei, S&P, etc.) but also single-stock options markets, interest-rate options markets, currency options markets, credit derivatives markets, and almost any other volatility market. New options markets often begin with traders using the canonical Black-Scholes model with no smile; then, as they gain experience with the idiosyncrasies of their particular markets and its movements, the implied volatility smile structure starts to develop.

The industry-standard Black-Scholes model alone cannot account for this structure, and so, though options prices are still quoted by means of their Black-Scholes implied volatilities, trading
desks at hedge funds and investment banks now use more complex smile-consistent models to value and hedge their options.

What’s the right replacement model for Black-Scholes? Think a little about how you would determine this. Or, it might be even better to ask, is there a right replacement model for Black-Scholes? That’s in part what this course is about, and there isn’t an easy answer.

This course will describe the smile structure of implied volatilities and the way that structure contradicts the classic Black-Scholes model. We’ll then consider some of the sorts of models that can account for the smile.

Why are these new models so important? Black-Scholes tells you that you can value an option because you can hedge away its risk. If Black-Scholes isn’t right, then you don’t know how to hedge the risk of options. Smile models are critical to the correct hedging of ordinary options and even more crucial for valuing exotic options and structured products that are very popular because they are custom-tailored for clients and generate higher margins. Because they are custom-tailored, these products are relatively illiquid, and, like custom-tailored clothes, can’t easily be resold. What are they therefore worth? If you can’t get a mark from another dealer, then only a model can tell you, and therefore their values are marked by model, and subject to model risk, or more accurately, to model uncertainty.

Quantitative strategists and financial engineers on derivatives trading desks and within the firm-wide risk departments at investment banks have to worry about which is the best model to use. There are many important issues of model choice, model validity and model testing that are of practical concern. Quants and controllers must worry about the marks of positions, to what extent they are model-dependent, what the effect a change in model has on the P&L of the firm. These are weighty issues that involve many people in the front and back office, and in I.T. groups. For example, it is common to value an exotic option with a very fancy slow model when you first think about the deal, and then mark it again daily with a less accurate and less sophisticated off-the-shelf model, because the slow model may use Monte Carlo simulation and take too much time to run. The question of model uncertainty also generates interesting and relevant questions about how to determine your profit and how to pay your traders for profits that depend on models and are therefore uncertain.

I’ll try to approach the models with a mixture of theory and pragmatism. I don’t like simply writing down formulas without proofs, though I’ll do that occasionally. Knowing the formulas like a table of integrals isn’t enough when you’re working in the field, because many of the derivative products you have to deal with, and their markets, violate the assumptions behind the simple formulas. So while you should know the standard models, and know how to derive and play around with them, you should aim to learn how to build your own models and understand what they lead to. You need to develop intuition about the models, so that you can know when your calculations are giving you the right answer or when you’ve made a mathematical or computational or programming mistake.

So, I’ll put a lot of effort into deriving simple or approximate proofs of the key model formulas and ideas. Often these proofs and formulas may not be the best way to implement a model for
rapid and accurate computational use, but they can be good for understanding ideas. My aim is to develop these models logically, to get a feel for the phenomena to be explained, and to estimate the effects of the models. I will also bring in two or three practitioners or traders from derivatives desks on Wall Street or at hedge funds to give talks about their parts of the business as it relates to options pricing, options trading and the volatility smile.
References

I will recommend papers to read along the way, but mostly my course is self-contained. I don’t require you to buy any textbook for the course but I can recommend the following for some additional material.


Some other useful books on the volatility smile:

- There is also a qualitative chapter (Chapter 14) on the smile in my book, *My Life as a Quant: Reflections on Physics and Finance*. I will post an electronic version of that chapter on Courseworks.

Some more useful books of a more general nature are listed below. All of them have sections on the volatility smile. The list isn’t comprehensive; there are so many others I don’t know about or haven’t mentioned:

- *Paul Wilmott on Quantitative Finance*, Wiley, by Paul Wilmott (who else?) is a very good general book on options theory. He’s not afraid to tell you what he thinks is important and what isn’t, which is valuable. (Actually, he’s not afraid to tell you what he thinks in general.) This is a good book in which to look up topics – you don’t have to read it cover to cover, but can dip in. And it’s always sensible.
- *Introduction to Quantitative Finance*, Stephen Blyth, 2013, is a very elegant and compact book with a straightforward approach that tries to make things simple and clear rather than complicated and obscure.
In terms of journals to read, look for example at

- Risk Magazine
- Wilmott Magazine
- Journal of Derivatives
- Quantitative Finance, lots of econophysics and quant papers
- www.ssrn.com has many papers in the FEN (Financial Economics Network) section, and most of the latest papers get posted there before publication.

I’ll give further references during the course. But mostly I will rely on my class lectures, which I will post on Courseworks.

Contacting me

You can email me at ed2114@columbia.edu.
If you have questions, please come to my office hours, which I will post. I cannot answer very long questions before or after class, or by email.

Grades

I will give homework assignments weekly. I will also give occasional pop quizzes in class for about 10 minutes at a time, unannounced. These, together with the homework sets, will count for 10% of the final grade. A further 40% will depend on the midterm and 50% on the final examination.

Homework is due once a week, on a Monday at the beginning of each class, to be left in a pile at the front of the classroom. The graded homeworks assigned two weeks earlier will be distributed in class too, at the break.

Ethics

This is important.

I don’t mind if you discuss homework among yourselves, but then I expect you to think about it and work it out and write it up by yourself afterwards. Please don’t hand be part of several people handing in identical copies of solutions; don’t hand in xeroxes of someone else’s solution. If you do, I will consider it cheating.

1. Everyone is an adult in this course, here to learn and behave responsibly.
2. Please don’t read email or browse the web in class. I have a policy of NO PHONES, NO LAPTOPS OPEN DURING CLASS.
3. If you have some emergency that requires you to keep your phone on, please let me know.
4. Please don’t come late to class.
5. In the past couple of years there has been some cheating on homework as well as on exams, in many FE courses. I don’t want to be a judge or jury. If someone is found cheating in any way, I will simply send their name to the dean’s office and let them take appropriate action.
Course Outline

This is roughly what I would like to cover in the course, but we will have to play it by ear and see exactly how things progress.

• **The Principles of Financial Modeling**
  Aim of the course.
  A quick look at the smile.
  Viewpoints: relative rather than absolute valuation
  The foundations of financial theory
  Valuation: Static hedging, Dynamic hedging, Utility-based
  The theory of dynamic hedging.

• **Option Valuation: Realities and Myths**
  The theory of dynamic replication
  Option replication
  The Black-Scholes equation
  P&L (profit and loss) of options trading
  The difficulties of dynamic hedging; which hedge ratio to use
  The approximations and assumptions involved
  Simulations of discrete hedging
  Reserves for illiquid securities

• **Introduction to the Implied Volatility Smile**
  The smile in various markets
  The difficulties the smile presents for trading desks and for theorists
  Pricing and hedging
  Different kinds of volatility
  Parametrizing options prices: delta, strike and their relationship
  Estimating the effects of the smile on delta and on exotic options
  Reasons for a smile
  No-riskless-arbitrage bounds on the size of the smile
  Fitting the smile
  Some simple models and a look at their smiles

• **Implied Distributions Extracted from the Smile**
  Arrow-Debreu state prices
  Breeden Litzenberger formula
  Black-Scholes implied density and its use
  Static replication of path-independent exotic options with vanilla options

• **Static Hedging**
  Static replication of path-dependent exotic options with vanilla options
• Extending Black-Scholes beyond constant-volatility lognormal stock price evolution
  Binomial trees
  Time-dependent deterministic rates
  Time-dependent deterministic volatility
  Calibration to rates and volatility
  Changes of numeraire to simplify problems
  Alternative stochastic processes that could account for the smile

• Local Volatility Models/ Implied Trees
  Derman-Kani binomial local volatility trees
  Difficulties encountered
  Trinomial local volatility trees

• Fitting Implied Binomial Trees to the Volatility Smile
  Dupire equation
  Fokker-Planck/ forward Kolmogorov equation.
  Calibration of implied binomial trees
  How to build an implied tree from options prices.
  The relation between local and implied volatilities

• The Consequences of Local Volatility Models
  The local volatility surface
  The relationship between local and implied volatility
  Estimating the deltas of vanilla options in the presence of the smile
  Estimating the values of exotic options
  Static hedging of barrier options
  Some specific local volatility models: displaced diffusion, CEV, mixed distributions

• Model classification
  Empirical behavior of implied volatility with time and market level
  Sticky strike, sticky delta, sticky implied tree

• Stochastic Volatility Models
  The effect of changes in volatility in the Black-Scholes formula
  The Vanna-Volga way of looking at things
  Mean reversion of volatility
  The SABR model
  The PDE for option value under stochastic volatility
  The mixing formula for option value under stochastic volatility
  Estimating the smile in stochastic volatility models
  Simulations of the smile in these models
  The relationship between local and stochastic volatility
• **Jump-Diffusion Models**
  Are they reasonable, and if so, when?
  Poisson jumps
  The Merton jump-diffusion model and its solution
  Estimating the smile in jump-diffusion models
  Simulations

• **Other Models**

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• **Some Guest Speakers**