Market Microstructure Invariants

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Overview

Our goal is to explain how order size, order frequency, market impact and bid-ask spread vary across stocks with different trading activity.

- We develop a model of market microstructure invariance that generates predictions concerning cross-sectional variations of these variables.

- These predictions are tested using a data set of portfolio transitions and find a strong support in the data.

- The model implies simple formulas for order size, order frequency, market impact, and bid-ask spread as functions of observable dollar trading volume and volatility.
A Framework

When portfolio managers trade stocks, they can be thought of as playing trading games. Since managers trade many different stocks, we can think of them as playing many different trading games simultaneously, a different game for each stock.

The intuition behind a trading game was first described by Jack Treynor (1971). In that game informed traders, noise traders and market makers traded with each other.
Games Across Stocks

Stocks are different in terms of their trading activity: dollar trading volume, volatility etc. Trading games look different across stocks only at first sight!

Our intuition is that trading games are the same across stocks, except for the length of time over which these games are played or the speed with which they are played.

“Business time” passes faster for more actively traded stocks.
Games Across Stocks

Only the speed with which business time passes varies as trading activity varies:

- **For active stocks** (high trading volume and high volatility), trading games are played at a **fast pace**, i.e. the length of trading day is small and business time passes quickly.

- **For inactive stocks** (low trading volume and low volatility), trading games are played at a **slow pace**, i.e. the length of trading day is large and business time passes slowly.
Reduced Form Approach

As a rough approximation, we assume that orders arrive according to a compound Poisson process with \textit{order arrival rate} $\gamma$ and \textit{order size} having a distribution represented by a random variable $\tilde{Q}$.

Both $\tilde{Q}$ and $\gamma$ vary across stocks.

The arrival rate $\gamma$, which measures market “velocity,” is proportional to the speed with which business time passes.
Bets

We think of orders as bets whose size is measured by dollar standard deviation over time.

Bet size over a calendar day:

\[ \tilde{B} = P \cdot \tilde{Q} \cdot \sigma \]

Bet size \( \tilde{B} \) measures the standard deviation of the mark-to-market gains per calendar day, conditional on number of shares \( \tilde{Q} \). Bet size increases as a square root with time.
Volatility in Business Time

Let $\sigma_0$ denote returns volatility in business time:

$$\sigma_0 = \sigma / \gamma^{1/2}$$

Bet size can be written

$$\tilde{B} = P \cdot \tilde{Q} \cdot \sigma_0 \cdot \gamma^{1/2}$$

Bet size is proportional to the square root of the rate $\gamma$ at which business time passes.
Trading Game Invariance

“Trading game invariance” is the hypothesis that bet size is constant when measured in units of business time, i.e., the distribution of the random variable

$$\tilde{I} := \tilde{B} \cdot \gamma^{-1/2} = P \cdot \tilde{Q} \cdot \sigma_0$$

does not vary across stocks or across time. Bet risk in calendar time remains proportional to the square root of the rate $\gamma$ at which business time passes:

$$\tilde{B} = \gamma^{1/2} \cdot \tilde{I}$$
Trading Activity

Stocks differ in their "Trading Activity" $W$, or a measure of gross risk transfer, defined as dollar volume adjusted for volatility $\sigma$:

$$W = V \cdot P \cdot \sigma = \zeta/2 \cdot \gamma \cdot E\{|\tilde{B}|\}.$$ 

Execution of bets induces extra volume; $\zeta$ adjusts for non-bet volume; we might assume $\zeta$ is constant and equal to two.
Key Result

Trading game invariance implies trading activity is proportional to $\gamma^{3/2}$:

$$W = \xi/2 \cdot \gamma^{3/2} \cdot E\{\tilde{I}\}.$$ 

Therefore

$$\gamma \propto W^{2/3}$$

and

$$\tilde{B} \propto W^{1/3} \cdot \tilde{I}$$
Theoretical Irrelevance Principles

The following irrelevance principles are satisfied by trading our invariance assumptions.

**Modigliani-Miller Irrelevance:** The trading game involving a financial security issued by a firm is \textit{independent} of its \textit{capital structure}:

- Stock Split Irrelevance,
- Leverage Irrelevance.

**Time-Clock Irrelevance:** The trading game is \textit{independent} of the speed at which the \textit{time clock} ticks.
Price Impact and Spread

Irrelevancies have implications for price impact and bid-ask spread, but we need to make additional assumptions which are consistent with many models:

- Linear price impact of bets leads to a fraction $\psi^2$ of price variance.
  Equivalent to assuming that the expected price impact cost of a bet is some constant $C_L$.

- Bid-ask spread cost of a bet is a fraction $\phi$ of price impact cost of a bet.
  Equivalent to assuming that the expect bid-ask spread cost of a bet is some constant $C_K$. 
Market Microstructure Invariance as an Empirical Hypothesis

For different securities and the same securities at different times:

- **Trading Game Invariance**: The distributions of trading game invariants \( \tilde{I} = \frac{\tilde{B}}{\gamma^{1/2}} \) are the same.

- **Market Impact Invariance**: Linear price impact of bets explains the same constant fraction \( \psi^2 \) of returns variance.

- **Bid-Ask Spread Invariance**: Bid-ask spread cost of a bet is the same constant fraction \( \phi \) of impact costs of a bet.

Market Microstructure Invariants: \( \tilde{I}, \psi, \) and \( \phi \).
Invariance Assumptions Identify Market Depth

Kyle (1985) and other models imply a linear price impact formula

\[ \lambda = \frac{\sigma_V}{\sigma_U} \]

where \( \sigma_V \) is the standard deviation of dollar price change per share resulting from price impact, and \( \sigma_U \) is the standard deviation of “order imbalances”.

- Trading game invariance identifies \( \sigma_U \):
  \[ \sigma_U = (\gamma \cdot E\{\tilde{Q}^2\})^{1/2} \]

- Market depth invariance identifies \( \sigma_V \):
  \[ \sigma_V = \psi \cdot \sigma \cdot P \]
Invariance and Transaction Costs

Trading game invariance $\tilde{I}$, market impact invariance, and bid-ask spread invariance make it possible to express price impact $\lambda$ and spread $\kappa$ as functions of trading activity $W$ and invariance constants:

\[
\frac{\lambda \cdot V}{\sigma \cdot P} = \psi \cdot \zeta^{2/3} \cdot \frac{\text{E}\{|\tilde{I}|\}^{2/3}}{\text{E}\{\tilde{I}^2\}^{1/2}} \cdot W^{1/3}.
\]

\[
\frac{\kappa}{\sigma \cdot P} = \frac{1}{2} \cdot \phi \cdot \psi \cdot \zeta^{1/3} \cdot \frac{\text{E}\{\tilde{I}^2\}^{1/2}}{\text{E}\{|\tilde{I}|\}^{2/3}} \cdot W^{-1/3}.
\]
A Benchmark Stock

**Benchmark Stock** - daily volatility $\sigma = 200$ bps, price $P^* = $40, volume $V^* = 1$ million shares. Trades over a calendar day:

Arrival Rate $\gamma^* = 4$

Avg. Order Size $\tilde{Q}^*$ as fraction of $V^* = 1/4$

Market Impact of $1/4 V^* = 200$ bps / $4^{1/2} = 100$ bps

Spread $= k$ bps
**Market Microstructure Invariance - Intuition**

**Benchmark Stock with Volume** $V^*$

$(\gamma^*, \tilde{Q}^*)$

**Stock with Volume** $V = 8 \cdot V^*$

$(\gamma = \gamma^* \cdot 4, \tilde{Q} = \tilde{Q}^* \cdot 2)$

**Avg. Order Size** $\tilde{Q}^*$ as fraction of $V^*$

$= 1/4$

**Market Impact of 1/4 $V^*$**

$= 200 \text{ bps} / 4^{1/2} = 100 \text{ bps}$

**Market Impact of 1/16 $V$**

$= 200 \text{ bps} / (4 \cdot 8^{2/3})^{1/2} = 50 \text{ bps}$

**Avg. Order Size** $\tilde{Q}$ as fraction of $V$

$= 1/16 = 1/4 \cdot 8^{-2/3}$

**Market Impact of 1/4 $V$**

$= 4 \cdot 50 \text{ bps} = 100 \text{ bps} \cdot 8^{1/3}$

**Spread**

$= k \text{ bps} \cdot 8^{-1/3}$
Market Microstructure Invariance - Predictions

If trading activity $W$ increases by one percent, some algebra implies the following cross-sectional predictions:

- **Trade Size**, as a percentage of average daily volume, decreases by 2/3 of one percent;
- **Market impact** of trading $X/V$ percent of average daily volume increases by 1/3 of one percent;
- **Bid-ask spread** decreases by 1/3 of one percent.
Market Microstructure Invariance - Math

The Model of Market Microstructure Invariance implies:

- **Market Impact**: \( \lambda_{TG} = \text{const} \cdot W^{1/3} \cdot \frac{\sigma P}{V} \)
- **Bid-Ask Spread**: \( k_{TG} = \text{const} \cdot W^{-1/3} \cdot \sigma P \)
- **Order Size**: \( \frac{|\tilde{Q}_{TG}|}{V} = \text{const} \cdot |\tilde{B}_H| \cdot W^{-2/3} \)
- **Length of Trading Day**: \( H = \text{const} \cdot W^{-2/3} \)

where \( W = V \cdot P \cdot \sigma \) defines trading activity.
**Alternative Theories**

We consider two alternative theories:

1. Naive alternative **Model of “Invariant Bet Frequency”** based on intuition that as trading activity increases, the size of bets increases proportionally, but their arrival rate remains constant.

2. Naive alternative **Model of “Invariant Bet Size”** based on the intuition that as trading activity increases, the size of bets remains the same, but their arrival rate increases proportionally.
Model of Invariant Bet Frequency

Model of Invariant Bet Frequency assumes that all variation in trading activity $W$ is explained entirely by variation in bet size.

As trading activity varies across stocks,

- Bet size $\tilde{B}$ varies proportionally.
- Bet frequency $\gamma$ remains constant.
Invariant Bet Frequency - Intuition

**Benchmark Stock with Volume** \( V^* \)

\((\gamma^*, \tilde{Q}^*)\)

- **One CALENDAR Day**
  - **buy orders**
  - **sell orders**

**Avg. Order Size** \( \tilde{Q}^* \) as fraction of \( V^* \) = 1/4

**Market Impact of** \( 1/4 \) \( V^* \)

= 200 bps / \( 4^{1/2} \) = 100 bps

**Spread**

= \( k \) bps

---

**Stock with Volume** \( V = 8 \cdot V^* \)

\((\gamma = \gamma^*, \tilde{Q} = \tilde{Q}^* \cdot 8)\)

- **One CALENDAR Day**
  - **buy orders**
  - **sell orders**

**Avg. Order Size** \( \tilde{Q} \) as fraction of \( V \) = 1/4

**Market Impact of** \( 1/4 \) \( V \)

= 200 bps / \( 4^{1/2} \) = 100 bps

**Spread**

= \( k \) bps
Invariant Bet Frequency - Predictions

If trading activity increases by one percent, then some math implies the following cross-sectional predictions:

- **Trade Size**, as a fraction of average daily volume, is constant because order size increases proportionally with average daily volume;

- **Market impact** of trading $X$ percent of average daily volume is constant;

- **Bid-ask spread** is constant.

**Intuition:** There is the same number of independent (but larger) bets per day, trading volume and order imbalances increase at the same rate; market depth does not change.
Invariant Bet Frequency - Comment

We believe that the model of Invariant Bet Frequency is the “default model” that implicitly but incorrectly guides the intuition of many asset managers.

- Model justifies trading say no more than 1% of average daily volume for all stocks, regardless of level of trading activity.

- Model justifies imputing same number of basis points in transactions costs for individual stocks in a basket with both active and inactive stocks, where size of trades are proportional to average daily volume.
Invariant Bet Frequency - Prediction Math

The Model of Invariant Bet Frequency implies:

- **Market Impact**: \( \lambda_\gamma = \text{const} \cdot W^0 \cdot \frac{\sigma P}{V} \)
- **Bid-Ask Spread**: \( k_\gamma = \text{const} \cdot W^0 \cdot \sigma P \)
- **Order Size**: \( \frac{|\tilde{Q}_\gamma|}{V} = \text{const} \cdot |\tilde{Z}| \cdot W^0 \)
- **Length of Trading Day**: \( H = 1 \cdot W^0 \)

where \( W = V \cdot P \cdot \sigma \) defines trading activity.
Model of Invariant Bet Size

Model of **Invariant Bet Size** assumes that all variation in trading activity is explained exclusively by variation in **bet frequency**.

As trading activity varies across stocks,

- Bet size $\tilde{B}$ remains **constant**.
- Bet frequency $\gamma$ **varies** proportionally.
Invariant Bet Size - Intuition

Benchmark Stock with Volume $V^*$

$\left( \gamma^*, \tilde{Q}^* \right)$

Stock with Volume $V = 8 \cdot V^*$

$\left( \gamma = \gamma^* \cdot 8, \quad \tilde{Q} = \tilde{Q}^* \right)$

Avg. Order Size $\tilde{Q}^*$ as fraction of $V^*$

$= 1/4$

Market Impact of $1/4 \ V^*$

$= 200 \text{ bps} / 4^{1/2} = 100 \text{ bps}$

Spread

$= k \text{ bps}$

Market Impact of $1/32 \ V$

$= 200 \text{ bps} / 32^{1/2}$

Avg. Order Size $\tilde{Q}$ as fraction of $V$

$= 1/32 = 1/4 \cdot 8^{-1}$

Market Impact of $1/4 \ V$

$= 8 \cdot 200 \text{ bps} / 32^{1/2} \text{ bps} = 100 \text{ bps} \cdot 8^{1/2}$

Spread

$= k \text{ bps} \cdot 8^{-1/2}$
Invariant Bet Size - Predictions

If trading activity increases by one percent, then some math implies the following cross-sectional predictions:

- **Trade Size**, as a fraction of average daily volume, decreases by one percent;
- **Market impact** of trading \( X \) percent of average daily volume increase by \( \frac{1}{2} \) of one percent;
- **Bid-ask spread** decreases by \( \frac{1}{2} \) of one percent.

**Intuition:** Since there are more independent bets per day, trading volume increases twice as fast as order imbalances. Thus, market depth increases at half the rate as trading volume.
Invariant Bet Size - Prediction Math

The Model of Invariant Bet Size implies:

- **Market Impact**: $\lambda_B = \text{const} \cdot W^{1/2} \cdot \frac{\sigma_P}{V}$,
- **Bid-Ask Spread**: $k_B = \text{const} \cdot W^{-1/2} \cdot \sigma_P$,
- **Order Size**: $|\tilde{Q}_B| = \text{const} \cdot |\tilde{B}| \cdot W^{-1}$,
- **Length of Trading Day**: $H = 1 \cdot W^0$,

where $W = V \cdot P \cdot \sigma$ defines trading activity.
Identification

- Note that the level of market impact, the level of bid-ask spreads, and the average size of bets are not identified by the theory, but can be estimated from data.

- Note that the length of the trading day is not identified as well, and we cannot estimate it from data using our methodology either.
The Model of Market Microstructure Invariance:
(Trading Game + Market Impact + Bid-Ask Spread Invariance)

Market Microstructure Invariants: $\tilde{I}$, $\psi$, and $\phi$. 
Testing - Portfolio Transition Data

The empirical implications of the three proposed models are tested using a proprietary dataset of portfolio transitions.

- Portfolio transition occurs when an old (legacy) portfolio is replaced with a new (target) portfolio during replacement of fund management or changes in asset allocation.

- Our data includes 2,680+ portfolio transitions executed by a large vendor of portfolio transition services over the period from 2001 to 2005.

- Dataset reports executions of 400,000+ orders with average size of about 4% of ADV.
Portfolio Transitions and Trades

We use the data on transition orders to examine which model makes the most reasonable assumptions about how the size of trades varies with trading activity.
Distribution of Order Sizes

Trading game invariance predicts that distributions of order sizes $X$, adjusted for differences in trading activity $W$, are the same across different stocks:

$$\ln \left( \frac{\tilde{Q}}{V} \cdot \left[ \frac{W}{W^*} \right]^{-2/3} \right).$$

We compare these distributions across 10 volume and 5 volatility groups.
Trading game invariance works well for **entire distributions** of order sizes. These distributions are approximately **log-normal**.
Log-Normality of Order Size Distributions


Panel B: Logarithm of Ranks against Quantiles of Empirical Distribution.

Trading game invariance works well for entire distributions of order sizes. These distributions are approximately log-normal.
Tests for Orders Size - Design

All three models are nested into one specification that relates trading activity $W$ and the trade size $\tilde{Q}$, proxied by a transition order of $X$ shares, as a fraction of average daily volume $V$:

$$\ln \left[ \frac{X_i}{V_i} \right] = \bar{q} + a_0 \cdot \ln \left[ \frac{W_i}{W_\star} \right] + \tilde{\epsilon}$$

The variables are scaled so that $e^{\bar{q}} \cdot 10^4$ is (assuming log-normal distribution) the median size of liquidity trade as a fraction of daily volume (in bps) for a benchmark stock with:

- daily standard deviation of 2%,
- price of $40$ per share,
- trading volume of 1 million shares per day,
- trading activity $W_\star = 2\% \cdot $40 \cdot 1 million.
Tests for Orders Size - Design

Three models differ only in their predictions about parameter $a_0$.

- **Model of Trading Game Invariance:** $a_0 = -2/3$.
- Model of Invariant Bet Frequency: $a_0 = 0$.
- Model of Invariant Bet Size: $a_0 = -1$.

We estimate the parameter $a_0$ to examine which of three models make the most reasonable assumptions.
Tests for Order Size: Results

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<th>All</th>
<th>NYSE</th>
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<th>NASDAQ</th>
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<td></td>
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<td></td>
<td><strong>Buy</strong></td>
<td><strong>Sell</strong></td>
<td><strong>Buy</strong></td>
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<tr>
<td>$\bar{q}$</td>
<td>-5.67***</td>
<td>-5.68***</td>
<td>-5.63***</td>
<td>-5.75***</td>
<td>-5.65***</td>
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<tr>
<td></td>
<td></td>
<td>(0.017)</td>
<td>(0.022)</td>
<td>(0.018)</td>
<td>(0.033)</td>
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<tr>
<td>$a_0$</td>
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<td>-0.63***</td>
<td>-0.60***</td>
<td>-0.71***</td>
<td>-0.61***</td>
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<tr>
<td></td>
<td></td>
<td>(0.008)</td>
<td>(0.010)</td>
<td>(0.008)</td>
<td>(0.019)</td>
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</table>

Model of Trading Game Invariance: $a_0 = -2/3$.

Model of Invariant Bet Frequency: $a_0 = 0$.

Model of Invariant Bet Size: $a_0 = -1$.

*** is 1%-significance, ** is 5%-significance, * is 10%-significance.
Why Coefficients for Sells Different from Buys

Since asset managers are “long only,” buys are related to current value of $W$, while sells are related to value of $W$ when stocks were bought.

Since increases in $W$ result from positive returns, higher values of $W$ are correlated with higher past returns.

Implies sell coefficients smaller in absolute value than buy coefficients, consistent with empirical results.

Adding lagged returns or lagged trading activity $W$ may improve results.
### Tests for Orders Size - F-Tests

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<th>NYSE</th>
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<td></td>
<td>All</td>
<td>Buy</td>
<td>Sell</td>
<td>Buy</td>
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<td>Model of Trading Game Invariance: $a_0 = -2/3$</td>
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<td></td>
<td></td>
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<tr>
<td>F-test</td>
<td>17.03</td>
<td>13.74</td>
<td>72.00</td>
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<tr>
<td>p-val</td>
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<tr>
<td>F-test</td>
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## Tests for Orders Size - $R^2$

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<td></td>
<td>Buy</td>
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</tr>
<tr>
<td><strong>Three Parameters: $P$, $V$, $\sigma$</strong></td>
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<tr>
<td>$Adj. R^2$</td>
<td>0.3211</td>
<td>0.2614</td>
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<tr>
<td><strong>One Parameter: $W = P \cdot V \cdot \sigma$</strong></td>
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<td>$Adj. R^2$</td>
<td>0.3188</td>
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<td>$Adj. R^2$</td>
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<td>-0.0002</td>
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<td><strong>Model of Invariant Bet Size: $a_0 = -1$</strong></td>
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<tr>
<td>$Adj. R^2$</td>
<td>0.2105</td>
<td>0.1683</td>
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</table>
Tests for Orders Size - Summary

Model of Trading Game Invariance assumes: An increase of one percent in trading activity $W$ leads to a decrease of $2/3$ of one percent in size of liquidity trade as a fraction of daily volume (for constant returns volatility).

Results: The estimates provide strong support for Model of Trading Game Invariance. The coefficient predicted to be $-2/3$ is estimated to be $-0.63$.

Discussion:

- The assumptions made in our model match the data economically.
- F-test rejects our model statistically because of small standard errors.
- Alternative models are rejected soundly with very large F-values.
- Estimating coefficients on $P$, $V$, $\sigma$ improves $R^2$ very little compared with imposing coefficient value of $-2/3$. 
Order Sizes Across Volume Groups

Do the data match models’ assumptions across 10 volume groups?

\[
\ln \left( \frac{X_i}{V_{1,i}} \right) = \left[ \sum_{j=1}^{10} I_{j,i} q_j \right] + a_0 \cdot \ln \left( \frac{W_i}{W_*} \right) + \tilde{\epsilon}
\]

- **Parameter** $a_0$ is restricted to values predicted by each model ($a_0 = -2/3$, $a_0 = 0$, or $a_0 = -1$).

- **Indicator variable** $I_{j,i}$ is one if $i$th order is in the $j$th volume groups.

- **Dummy variables** $\bar{q}_j, j = 1, \ldots, 10$ quantify the average trade size for a benchmark stock based on data for $j$th volume group. If assumptions of the model are reasonable, then all dummy variables should be constant across volume groups.
Average Order Sizes Across Volume Groups

Figure plots average order size $\bar{q}_j$ across 10 volume groups. Group 1 contains stocks with the lowest volume. Group 10 contains stocks with the highest volume. The volume thresholds are 30th, 50th, 60th, 70th, 75th, 80th, 85th, 90th, and 95th percentiles for NYSE stocks.
Tests for Orders Size - Summary

**Predictions:** If the data match assumptions well, then all dummy variables $\tilde{q}_j, j = 1, .10$ should be constant across volume groups.

**Results:** The data match the assumptions of Model of Trading Game Invariance much better than the two alternative models.

**Discussion:**
- Pattern of dummy variables of Model of Trading Game Invariance is reasonably constant.
- But note that in Model of Trading Game Invariance, trade size for largest 5% of stocks is statistically larger than predicted by the model, due to low standard errors.
- Alternative models fail miserably to explain the data on trade sizes.
Tests for Orders Size - Conclusion

Data on the sizes of portfolio transition orders strongly support assumptions made in Model of Trading Game Invariance. The data soundly reject assumptions made in alternative models.

Intuition: when trading activity increases, both frequency and size of trades increase; neither remains constant.
Portfolio Transitions and Trading Costs

We use data on the implementation shortfall of portfolio transition trades to test predictions of the three proposed models concerning how transaction costs, both market impact and bid-ask spread, vary with trading activity.
Portfolio Transitions and Trading Costs

“Implementation shortfall” is the difference between actual trading prices (average execution prices) and hypothetical prices resulting from “paper trading” (price at previous close).

There are several problems usually associated with using implementation shortfall to estimate transactions costs. Portfolio transition orders avoid most of these problems.
Problem I with Implementation Shortfall

Implementation shortfall is a biased estimate of transaction costs when it is based on price changes and executed quantities, because these quantities themselves are often correlated with price changes in a manner which biases transactions costs estimates.

**Example A:** Orders are often canceled when price runs away. Since these non-executed, high-cost orders are left out of the sample, we would underestimate transaction costs.

**Example B:** When a trader places an order to buy stock, he has in mind placing another order to buy more stock a short time later.

For **portfolio transitions**, this problem does not occur: Orders are not canceled. The timing of transitions is somewhat exogenous.
Problems II with Implementation Shortfall

The second problem is statistical power.

**Example:** Suppose that 1% ADV has a transactions cost of 20 bps, but the stock has a volatility of 200 bps. Order adds only 1% to the variance of returns. A properly specified regression will have an R squared of 1% only!

For **portfolio transitions**, this problem does not occur: Large and numerous orders improve statistical precision.
Tests For Market Impact and Spread - Design

All three models are nested into one specification that relates trading activity $W$ and implementation shortfall $C$ for a transition order for $X$ shares:

$$C_i \cdot \left[ \frac{0.02}{\sigma} \right] = \frac{1}{2} \bar{\lambda} \cdot \left[ \frac{W_i}{W_*} \right]^{\alpha_0} \cdot \frac{X_i}{(0.01)V_i} + \frac{1}{2} \bar{k} \cdot \left[ \frac{W_i}{W_*} \right]^{\alpha_1} \cdot \frac{(X_{omt,i} + X_{ec,i})}{X_i} + \tilde{\epsilon}$$

The variables are scaled so that parameters $\bar{\lambda}$ and $\bar{k}$ measure in basis point the market impact (for 1% of daily volume $V$) and spread for a benchmark stock with volatility 2% per day, price $40$ per share, and daily volume of 1 million shares.

- Spread is assumed to be paid only on shares executed externally in open markets and external crossing networks, not on internal crosses.
- Implementation shortfall is adjusted for differences in volatility.
Tests For Market Impact and Spread - Design

The three models make different predictions about parameters $a_0$ and $a_1$.

- **Model of Trading Game Invariants:** $\alpha_0 = 1/3, \alpha_1 = -1/3$.
- **Model of Invariant Bet Frequency:** $\alpha_0 = 0, \alpha_1 = 0$.
- **Model of Invariant Bet Size:** $\alpha_0 = 1/2, \alpha_1 = -1/2$.

We estimate $a_0$ and $a_1$ to test which of three models make the most reasonable predictions.
Tests For Market Impact and Spread - Results

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
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<th>NASDAQ</th>
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<tbody>
<tr>
<td></td>
<td>All</td>
<td>Buy</td>
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<td></td>
</tr>
<tr>
<td>$\frac{1}{2} \bar{\lambda}$</td>
<td>2.85***</td>
<td>2.50***</td>
<td>2.33***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.245)</td>
<td>(0.515)</td>
<td>(0.365)</td>
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<tr>
<td>$\alpha_0$</td>
<td>0.33***</td>
<td>0.18***</td>
<td>0.33***</td>
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</tr>
<tr>
<td></td>
<td>(0.024)</td>
<td>(0.045)</td>
<td>(0.054)</td>
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</tr>
<tr>
<td>$\frac{1}{2} \bar{k}$</td>
<td>6.31***</td>
<td>14.99***</td>
<td>2.82*</td>
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<tr>
<td></td>
<td>(1.131)</td>
<td>(2.529)</td>
<td>(1.394)</td>
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<tr>
<td>$\alpha_1$</td>
<td>-0.39***</td>
<td>-0.19***</td>
<td>-0.46***</td>
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<tr>
<td></td>
<td>(0.025)</td>
<td>(0.045)</td>
<td>(0.061)</td>
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- Model of Trading Game Invariance: $\alpha_0 = 1/3, \alpha_1 = -1/3$.
- Model of Invariant Bet Frequency: $\alpha_0 = 0, \alpha_1 = 0$.
- Model of Invariant Bet Size: $\alpha_0 = 1/2, \alpha_1 = -1/2$.

*** is 1%-significance, ** is 5%-significance, * is 10%-significance.
### Tests For Market Impact and Spread - F-Tests

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
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<th>NYSE</th>
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<tr>
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<td>Sell</td>
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<td>Model of Trading Game Invariance:</td>
<td>$\alpha_0 = 1/3, \alpha_1 = -1/3$</td>
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<tr>
<td>F-test</td>
<td>2.60</td>
<td>8.57</td>
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<td>p-val</td>
<td>0.0742</td>
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<tr>
<td>Model of Invariant Bet Frequency:</td>
<td>$\alpha_0 = 0, \alpha_1 = 0$</td>
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<tr>
<td>F-test</td>
<td>176.14</td>
<td>14.77</td>
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<td>0.0000</td>
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<tr>
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<tr>
<td>Model of Invariant Bet Size:</td>
<td>$\alpha_0 = 1/2, \alpha_1 = -1/2$</td>
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<tr>
<td>F-test</td>
<td>30.34</td>
<td>39.81</td>
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<td>p-val</td>
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Kyle and Obizhaeva  
Market Microstructure Invariants
Tests for Impact and Spread - Summary

Model of Trading Game Invariance predicts: The coefficient for market impact is $\alpha_0 = 1/3$. The coefficient for bid-ask spread is $\alpha_1 = -1/3$.

Results: Coefficient $\alpha_0$ is estimated to be $0.33$, matching prediction of Trading Game Invariance exactly. Coefficient $\alpha_1$ is estimated to be $-0.39$, matching prediction of the model reasonably closely.

Discussion:

- Model of Trading Game Invariance is statistically rejected due to small standard errors and imperfect match for spread.
- Alternative models are soundly rejected.
- For benchmark stock, half-spread is 7.90 basis points and half market impact is 2.89 basis points (restricting $\alpha_0$ to be 1/3 and $\alpha_1$ to be -1/3).
Transactions Costs Across Volume Groups

Do the data match models’ assumptions across 10 volume groups?

\[ C_i \cdot \left[ \frac{0.02}{\sigma} \right] = \left( \sum_{j=1}^{10} \Pi_{j,i} \cdot \frac{1}{2} \lambda_j \right) \left[ \frac{W_i}{W_*} \right]^{\alpha_0} \frac{X_i}{(0.01)V_i} + \left( \sum_{j=1}^{10} \Pi_{j,i} \cdot \frac{1}{2} k_j \right) \left[ \frac{W_i}{W_*} \right]^{\alpha_1} \frac{(X_{omt,i} + X_{ec,i})}{X_i} + \tilde{\epsilon} \]

- **Parameter** \( \alpha_0 \) **and** \( \alpha_1 \) are restricted to values predicted by each model
  - \( \alpha_0 = 1/3, \alpha_0 = -1/3; \alpha_0 = 0, \alpha_0 = 0; \) or \( \alpha_0 = 1/2, \alpha_0 = -1/2 \).

- **Indicator variable** \( \Pi_{j,i} \) is one if \( i \)th order is in the \( j \)th volume groups.

- **Dummy variables** \( \bar{\lambda}_j \) **and** \( \bar{k}_j, j = 1, \ldots 10 \) quantify the market impact and spread for \( j \)th volume group. If assumptions of the model are reasonable, then all dummy variables should be constant across volume groups.
Transactions Costs Across Volume Groups

Figure plots **half market impact** $\frac{1}{2} \bar{\lambda}_j$ and **half effective spread** $\frac{1}{2} \bar{k}_j$ across **10 volume groups**. Group 1 contains stocks with the lowest volume. Group 10 contains stocks with the highest volume. The volume thresholds are 30th, 50th, 60th, 70th, 75th, 80th, 85th, 90th, and 95th percentiles for NYSE stocks.
Tests for Impact and Spread - Summary

**Predictions:** If the data match predictions well, then all dummy variable $\bar{\lambda}_j$ and $\bar{k}_j, j = 1, ..10$ should be constant across volume groups.

**Results:** Pattern is more stable for our model of Trading Game Invariance than for other two models.

**Discussion:**

- High precision for small stock anchors models parameters.
- For model of Trading Game Invariance, most active stocks have less impact and higher spreads than predicted, due to basket trades?
- Model of Invariant Bet Frequency gives more weight to orders in small stocks (since these orders are large relative to volume) and incorrectly extrapolates the estimates for small stocks to large ones. This model does reasonably when small stocks are excluded from the sample.
Conclusions

Our tests provide strong support for the model of Trading Game Invariance which implies, for example, that a one percent increase in trading activity $W = V \cdot P \cdot \sigma$ is associated with ...

- an increase of $1/3$ of one percent in average order size,
- an increase of $2/3$ of one percent in its arrival frequency,

and leads to...

- an increase of $1/3$ of one percent in market impact,
- a decrease of $1/3$ of one percent in bid-ask spread.
Results Related to Quoted Spread

Regression of log of spread on log of trading activity $W$:

- Predicted coefficient is $-1/3$.
- Estimated coefficient is $-0.31$ for NYSE and $-0.40$ for NASDAQ.

Using quoted spread rather than implicit realized spread cost in transactions cost regression:

- Predicted coefficient is $2/3$.
- Estimated coefficient is $0.62$, but puzzling variation across NYSE and NASDAQ, buys and sells.
Calibration: Transactions Cost Formula

For a **benchmark stock**, half market impact $\frac{1}{2}\lambda^*$ is 2.89 basis points and half-spread $\frac{1}{2}k^*$ is 7.90 basis points.

The Model of Market Microstructure Invariants extrapolates these estimates and allows us to calculate **expected trading costs** for any order of $X$ shares for **any security** using a simple formula:

$$C(X) = \frac{1}{2} \lambda^* \left( \frac{W}{(40)(10^6)(0.02)} \right)^{1/3} \frac{\sigma}{0.02} \frac{X}{(0.01)V} + \frac{1}{2} k^* \left( \frac{W}{(40)(10^6)(0.02)} \right)^{-1/3} \frac{\sigma}{0.02},$$

where trading activity $W = \sigma \cdot P \cdot V$

- $\sigma$ is the expected daily volatility,
- $V$ is the expected daily trading volume in shares,
- $P$ is the price.
Calibration: Implications of Log-Normality for Volume and Volatility

Standard deviation of log of bet size is $2.50^{1/2}$.

- Implies a one standard deviation increase in bet size is a factor of about 4.85.
- Implies 50% of trading volume generated by largest 5.71% of bets.
- Implies 50% of returns variance generated by largest 0.08% of bets.
Calibration: Bet Size and Trading Activity

Benchmark stock has $40 million daily volume and 2% daily returns standard deviation. For the benchmark stock, empirical results imply:

- Average bet size is 0.34% of expected daily volume.
- Benchmark stock has about 85 bets per day.
- Median bet size is $136,000; average bet size is $472,000.
- Order imbalances are 38% of daily trading volume.
“Time change” is that idea that a larger than usual number of independent price fluctuations results from business time passing faster than calendar time.

- Clark (1973): Price changes result from log-normal with time-varying variance, implying finite variance to price changes.
- Microstructure invariance: Kurtosis in returns results from rare, very large bets, due to high variance of log-normal. Caveat: Large bets may be executed very slowly, e.g., over weeks.
- Econophysics: Gabaix et al. (2006); Farmer, Bouchard, Lillo (2009). Right tail of distribution might look like a power law.
Liquidity and Velocity

- “Velocity” $\propto \gamma$
- Cost of Converting Asset to Cash $1/L_\$ \propto (PV/\sigma^2)^{1/3}$
- Cost of Transferring a Risk $1/L_\sigma \propto W^{-1/3}$
Calibration: Direct Estimate of Market Impact

Using order size data but not execution price data, market impact can be calibrated directly from formula

\[ \lambda = \frac{\sigma_V}{\sigma_U} = \frac{\psi \sigma P}{\zeta/2 \cdot [\gamma E\{\tilde{Q}^2\}]^{1/2}} \]

using assumptions such as \( \zeta = 2 \) and \( \psi = 1 \).

(This is consistent with Kyle (1985) linear impact formula
\( \lambda = \sigma_V/\sigma_U \).)

- Under the assumptions \( \zeta = 2 \) and \( \psi = 1.10 \), the results are the same as estimates based on implementation shortfall.
Market Temperature

Derman (2002) defines market temperature $\chi$ as $\chi = \sigma \cdot \gamma^{1/2}$. Standard deviation of order imbalances is $P \cdot \sigma_U = [\gamma \cdot E\{\tilde{Q}^2\}]^{1/2}$.

- Product of temperature and order imbalances proportional to trading activity: $P\sigma_U \cdot \chi \propto W$
- Invariance implies temperature $\propto (PV)^{1/3}\sigma^{4/3}$.
- Invariance implies expected market impact cost of an order $\propto (PV)^{1/3}\sigma^{4/3}$.

Therefore invariance implies temperature proportional to market impact cost of an order.
1987 Stock Market Crash

Facts about 1987 stock market crash:

- **Trading volume** on October 19 was $40 billion ($20 billion futures plus $20 billion stock). Typical volume was lower (say $20 billion) but inflation makes 1987 dollar worth more than 2001-2005 dollar.

- **Volatility** during crash was extremely high, so 2% expected volatility per day might be reasonable.

- From Wednesday to Tuesday, **portfolio insurers** sold $14 Billion ($10 billion futures plus $4 billion stock).

- From Wednesday to Tuesday, **S&P 500 futures** declined from 312 to 185, a decline of 41% (including bad basis). **Dow** declined from 2500 to 1700, a decline of 32%.
1987 Stock Market Crash

Our market impact formula implies decline of

\[
2 \cdot 2.89 \cdot \left( \frac{40 \cdot 10^9}{40 \cdot 10^6} \right)^{1/3} \cdot \frac{0.02}{0.02} \cdot \frac{14/40}{0.01} = 20.23\%
\]

Our model suggests **portfolio insurance selling** had market impact of about 20%. Keep in mind that assumptions are approximations, so result is an approximation as well.
Fraud at Société Générale, January 2008

Facts about a fraud:

- From Jan 21 to Jan 23, a fraudulent position of Jérôme Kerviel had to be liquidated: €30 billion in STOXX50 futures, €18 billion in DAX futures, and €2 billion in FTSE futures.

- Trading volume was €61 billion in STOXX50 futures, €38 billion in DAX futures, and €8 billion in FTSE futures.

- Volatility was about 1% per day.

- Bank has reported exceptional losses of €6.4 billion, which were attributed to “adverse market movements” between Jan 21 and Jan 23.
Fraud at Société Générale, January 2008

Our market impact formula implies a total liquidation costs of about \( \mathbf{€3.60\ billion} \).

If we take into account losses on Kerviel’s position during the market’s decline between 31 Dec 2007 and 18 Jan 2008 (estimated to range between \( \mathbf{€2\ billion} \) and \( \mathbf{€4\ billion} \)), we conclude that our estimates are consistent with reported losses of \( \mathbf{€6.4\ billion} \).
The “Flash Crash” of May 6, 2010

News media report that a large trader sold 75,000 S&P 500 E-mini contracts. Prices fell approximately 300 bp during first part of day, then suddenly fell about 500 bp over 10 minutes, then rose about 500 bp over next 10 minutes.

- One contracts represents ownership of about $55,000 with S&P level of 1100.
- Typical contract volume was about 2 million contracts per day, or $110 billion (but much higher on May 6).
- Volatility was high due to European debt crisis; rough estimate is \( \sigma = 0.02 \)

Our market impact formula implies decline of

\[
2 \cdot 2.89 \cdot \left( \frac{110 \cdot 10^9}{40 \cdot 10^6} \right)^{1/3} \cdot \frac{0.02}{0.02} \cdot \frac{75 \cdot 10^3}{2 \cdot 10^6 \cdot 0.01} = 303.67 \text{bp}
\]

Flash crash research in progress by Kirilenko, Kyle, Tuzun, and Samadi.
More Practical Implications

- **Trading Rate:** If it is reasonable to restrict trading of the benchmark stock to say 1% of average daily volume, then a smaller percentage would be appropriate for more liquid stocks and a larger percentage would be appropriate for less liquid stocks.

- **Components of Trading Costs:** For orders of a given percentage of average daily volume, say 1%, bid-ask spread is a relatively larger component of transactions costs for less active stocks, and market impact is a relatively larger component of costs for more active stocks.

- **Comparison of Execution Quality:** When comparing execution quality across brokers specializing in stocks of different levels of trading activity, performance metrics should take account of nonlinearities documented in our paper.
Evidence From TAQ Dataset Before 2001

Trading game invariance seems to work in TAQ before 2001, subject to market frictions (Kyle, Obizhaeva and Tuzun (2010)).
Trading game invariance is **hard to test** in TAQ after 2001.
News Articles and Trading Game Invariance

Data on the number of Reuters news items $N$ is consistent with trading game invariance (Kyle, Obizhaeva, Ranjan, and Tuzun (2010)).

![Graphs showing trends in intercept, slope, and overdispersion over time.](image)
More Philosophical Implications

Trades and prices are not completely random. There are similar structures, i.e. “trading games”, in the trading data. Trading games are invariant across stocks and across time, except they are played at different pace. Invariance theory provides a consistent and operational framework for describing financial markets.