Pricing Asian Options under a General Jump Diffusion Model

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Abstract

We obtain a closed-form solution for the double-Laplace transform of Asian options under the hyper-exponential jump diffusion model (HEM). Similar results are only available previously in the special case of Black-Scholes model (BSM). Even in the case of BSM, our approach is simpler as we essentially use only the Itô’s formula and do not need more advanced results such as those of Bessel processes and Lamperti’s representation. Furthermore, our approach is more general as it applies to the HEM. As a by-product we also show that a well-known recursion relating to Asian options has a unique solution in a probabilistic sense. The double-Laplace transforms can be inverted numerically via a two-sided Euler inversion algorithm. Numerical results indicate that our pricing method is fast, stable, and accurate.

Area of review: Financial engineering.

1 Introduction

Asian options (or average options), whose payoffs depend on the arithmetic average of the underlying asset price over a pre-specified time period, are among the most popular path-dependent options traded in both exchanges and over-the-counter markets. The main difficulty in pricing Asian options is that the distribution of the average price may not have an analytical solution.

There is a large literature on Asian options under the Black-Scholes model (BSM). For example, approaches based on partial differential equations were given in Ingersoll [24], Rogers and Shi [35], Lewis [27], Dubois and Lelièvre [16], Zhang [48, 49]; Monte Carlo simulation techniques were discussed in Broadie and Glasserman [5], Glasserman [22] and Lapeyre and
Temam [26]; approximations based on moments were derived in Turnbull and Wakeman [43] and Milevsky and Posner [32]. Lower and upper bounds were given in Curran [14], Henderson et al. [23], and Thompson [42]. Five other approaches that are closely related to ours are (i) Linetsky [28] derived an elegant series expansion for Asian options via a one-dimensional affine diffusion. (ii) Vecer [44] proposed a change of numéraire approach and obtained a one-dimensional partial differential equation (PDE) which can be solved numerically in stable ways. (iii) Based on Bessel processes and Lamperti’s representation, in a celebrated paper Geman and Yor [21] provided an analytical solution of a single-Laplace transform of the Asian option price with respect to the maturity. For further theoretical results, see, e.g., Yor [47], Matsumoto and Yor [30, 31], Carr and Schröder [11] and Schröder [37]. Significant progresses have been made for the inversion of the single-Laplace transform even in the challenging case of low volatilities ($\leq 0.05$) in Shaw [38, 40]. Dewynne and Shaw [15] gave a simple derivation of the single-Laplace transform, and provided a matched asymptotic expansions, which performs well for extremely low volatilities. (iv) Dufresne [17, 18] obtained many interesting results and a Laguerre series expansion for both Asian and reciprocal Asian options by using the moments of the integral of geometric Brownian motion and the reciprocal of the integral. (v) Double-Laplace and Fourier-Laplace transforms were proposed in Fu et al. [19] and Fusai [20], respectively. For the differences between their methods and ours, see Sections 2 and 5.1.3.

Note that all the papers discussed above are within the Black-Scholes framework. There are only few papers for alternative models with jumps. For example, Albrecher [2] gave an approximation of Asian options under a Lévy process using moments; Carmona et al. [9] derived some theoretical representations for Asian options under some special Lévy processes; Vecer and Xu [46] gave some representations for Asian options under semi-martingale models via partial integro-differential equations (PIDEs); Bayraktar and Xing [4] proposed a numerical approach to Asian options under jump diffusion models by constructing a sequence of converging functions.

In this paper, we investigate the pricing of Asian options under the hyper-exponential jump
diffusion model (HEM) where the jump sizes have a hyper-exponential distribution, i.e., a mixture of a finite number of exponential distributions. For background on the HEM, see Levendorskii [29] and Cai and Kou [7]. The contribution of the current paper is three-fold.

(1) Even in the special case of BSM, our approach is simpler as we essentially use only the Itô’s formula and do not need more advanced results such as those of Bessel processes and Lamperti’s representation. See Section 3.

(2) Our approach is more general as it applies to the HEM; see Section 4. As a by-product we also show that under the HEM a well-known recursion relating to Asian options has a unique solution in a probabilistic sense, and the integral of the underlying asset price process at the exponential time has the same distribution as a combination of a sequence of independent gamma and beta random variables; see Section 4.1.

(3) The double-Laplace transform can be inverted numerically via a latest two-sided Euler inversion algorithm along with a scaling factor proposed in Petrella [34]. Numerical results indicate that our pricing algorithm is fast, stable, and highly accurate. Indeed, our pricing method is highly accurate compared with the benchmarks from the three existing pricing methods under the BSM: (i) Linetsky’s method, (ii) Vecer’s method, and (iii) Geman and Yor’s single-Laplace method via Shaw’s elegant Mathematica implementation. Moreover, our method performs well even for low volatilities, e.g., 0.05. See Section 5.

The rest of the paper is organized as follows. Section 2 contains a general formulation of the double-Laplace transform of Asian option prices. To illustrate our idea clearly, in Section 3 we concentrate on pricing Asian options under the BSM. In Section 4, we extend the results in Section 3 to the more general HEM. Section 5 is devoted to the implementation of our pricing algorithm via the latest two-sided, two-dimensional Euler inversion algorithm with a scaling factor. We analyze the algorithm’s accuracy, stability, and low-volatility performance by conducting a detailed comparison with other existing methods under the BSM. Most of the proofs are in an online supplement.
2 A General Result for the Double-Laplace Transform of the Asian Option Price

For simplicity, we shall focus on Asian call options, as Asian put options can be treated similarly. The payoff of a continuous Asian call option with a mature time $t$ and a fixed strike $K$ is $(\frac{S_0 A_t}{t} - K)^+$, where $A_t := \int_0^t e^{X(s)} ds$, $S(t)$ is the underlying asset price process with $S(0) = S_0$, and $X(t) := \log(S(t)/S(0))$ is the return process. Standard finance theory says that the Asian option price at time zero can be expressed as $P(t, k) := e^{-rt} E \left[ (\frac{S_0 A_t}{t} - K)^+ \right]$, where $E$ is expectation under a pricing probability measure $\mathbb{P}$. Under the BSM the measure $\mathbb{P}$ is the unique risk neutral measure; whereas under more general models $\mathbb{P}$ may be obtained in other ways, such as using utility functions or mean variance hedging arguments. For more details, see, e.g., Shreve [41].

A key component of our double-Laplace inversion method is a scaling factor $X > S_0$. More precisely, with $k := \ln(\frac{X}{K})$ we can rewrite the option price $P(t, k) = e^{-rt} E \left( \frac{S_0}{X} A_t - e^{-k} \right)^+$. Note that $k$ can be either positive or negative, so the Laplace transform w.r.t. $k$ will be two-sided. The scaling factor $X$ introduced by Petrella [34] is primarily for overcoming two difficulties involved in the two-sided Euler inversion algorithm, i.e., helping control the associated discretization errors and making the inversion occur at a reasonable point $k$. For more details, we refer to Petrella [34]. The interested readers can also see Cai, Kou and Liu [8], where they introduced and discussed in more detail a shift parameter for the two-sided Euler inversion algorithm, which is similar to but more general than the scaling factor. As a consequence of introduction of the scaling factor, the resulting inversion algorithm is accurate, fast, and stable even in the case of low volatility, e.g. $\sigma = 0.05$. For further discussion, see Section 5.

The following result presents an analytical representation for the double-Laplace transform of $f(t, k) := X E(\frac{S_0}{X} A_t - e^{-k})^+$ w.r.t. $t$ and $k$. Note that Theorem 2.1 is a quite general result, as it holds under not only the BSM but also other stochastic models, for example, the HEM. The result reduces the problem of pricing Asian options to the study of real moments of exponentially-stopped average $E[A_{\mu}^{\nu+1}]$. 

4
**Theorem 2.1.** Let $\mathcal{L}(\mu, \nu)$ be the double-Laplace transform of $f(t, k)$ w.r.t. $t$ and $k$, respectively.

More precisely, $\mathcal{L}(\mu, \nu) = \int_0^\infty \int_{-\infty}^\infty e^{-\mu t} e^{-\nu k} X E(S_0 \frac{A_t}{X} - e^{-k})^+ dk dt$. Then we have that
\[
\mathcal{L}(\mu, \nu) = \frac{X E[A_{T\mu}^{\nu+1}]}{\mu \nu (\nu + 1)} \left( \frac{S_0}{X} \right)^{\nu + 1}, \quad \mu > 0, \quad \nu > 0,
\]
where $A_{T\mu} = \int_0^{T\mu} e^{X(s)} ds$ and $T\mu$ is an exponential random variable with rate $\mu$ independent of $\{X(t) : t \geq 0\}$. Here $\mu > 0$ and $\nu > 0$ should satisfy $E[A_{T\mu}^{\nu+1}] < +\infty$.

**Proof:** Applying Fubini’s theorem yields
\[
\mathcal{L}(\mu, \nu) = X \int_0^\infty e^{-\mu t} E \left[ \int_{-\ln(S_0A_t/X)}^\infty e^{-\nu k} \left( \frac{S_0}{X} \frac{A_t}{X} - e^{-k} \right) dk \right] dt
\]
\[
= X \int_0^\infty e^{-\mu t} E \left[ \frac{S_0}{X} A_t \int_{-\ln(S_0A_t/X)}^\infty e^{-\nu k} dk - \int_{-\ln(S_0A_t/X)}^\infty e^{-(\nu+1)k} dk \right] dt
\]
\[
= X \int_0^\infty e^{-\mu t} E \left[ \frac{(S_0A_t/X)^{\nu+1}}{\nu} - \frac{(S_0A_t/X)^{\nu+1}}{\nu + 1} \right] dt
\]
\[
= X \int_0^\infty e^{-\mu t} \frac{E[A_{T\mu}^{\nu+1}]}{\nu (\nu + 1)} \left( \frac{S_0}{X} \right)^{\nu + 1} dt = \frac{X}{\mu \nu (\nu + 1)} \left( \frac{S_0}{X} \right)^{\nu + 1} \cdot E[A_{T\mu}^{\nu+1}],
\]
from which the proof is completed. \( \Box \)

The idea of taking Laplace transform w.r.t. the log-strike $\ln(K)$ dates back to the work by Carr and Madan [10]. Here we use the scaled log-strike $\ln(X/(Kt))$ instead, as suggested in Petrella [34]. The idea of using double-Laplace transforms to price Asian options goes to the paper by Fu et al. [19]. Their double-Laplace transform, however, is greatly different from ours for the following two reasons. First, the double-Laplace transform in Fu et al. [19] is for pricing Asian options only under the BSM; whereas ours is for both the BSM and the more general HEM. Second, their double-Laplace transform is taken w.r.t. the maturity $t$ and the strike $K$; whereas ours is w.r.t. the maturity $t$ and a scaled log-strike $k \equiv \ln(X/(Kt))$. As a result, our expression is in terms of the real moments of $A_{T\mu}$, i.e., $E[A_{T\mu}^{\nu+1}]$, while Fu et al. [19] basically used the moment generating function $E[\exp \{\lambda A_{T\mu}\}]$. Finally, our formulation leads to a different expression for the double-Laplace transform, which only involves gamma functions, while the solution of Fu et al. [19] under the BSM is in terms of hypergeometric functions.
3 Pricing Asian Options under the BSM

In this section we shall study Asian option pricing under the BSM via the double-Laplace transforms. More precisely, we would like to investigate the distribution of $A_{T\mu}$ so that we can compute $E[A_{T\mu}^{\nu+1}]$ explicitly and hence obtain analytical solutions for the double-Laplace transforms, thanks to Theorem 2.1.

3.1 Distribution of $A_{T\mu}$ under the BSM

The classical BSM postulates that under the risk-neutral measure $\mathbb{P}$, the return process $\{X(t) = \log(S(t)/S(0)) : t \geq 0\}$ is given by

$$X(t) = \left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t), \quad X(0) = 0,$$

where $r$ is the risk-free rate, $\sigma$ the volatility, and $\{W(t) : t \geq 0\}$ the standard Brownian motion. The infinitesimal generator of $\{S(t)\}$ is given by

$$Lf(s) = \frac{\sigma^2}{2}s^2f''(s) + rsf'(s)$$

for any twice continuously differentiable function $f(\cdot)$, and the Lévy exponent of $\{X(t)\}$ is

$$G(x) := \frac{E[e^{xX(t)}]}{t} = \frac{\sigma^2}{2}x^2 + \left(r - \frac{\sigma^2}{2}\right)x.$$  

Denote by $\alpha_1$ and $\alpha_2$ the two roots of the equation $G(x) = \mu(> 0)$ under the BSM. Then,

$$\alpha_1 = \frac{-\overline{\mu} + \sqrt{\overline{\mu}^2 + 2\overline{\sigma}}}{2} > 0, \quad \alpha_2 = \frac{-\overline{\mu} - \sqrt{\overline{\mu}^2 + 2\overline{\sigma}}}{2} < 0,$$

where $\overline{\mu} = \frac{4\mu}{\sigma^2}$ and $\overline{\sigma} = \frac{2r}{\sigma^2} - 1$.

Consider the following nonhomogeneous ordinary differential equation (ODE)

$$Ly(s) = (s + \mu)y(s) - \mu, \quad \text{for } s \geq 0.$$  

Note that the equation (5) has two singularities, a regular singularity at 0 and an irregular singularity at $+\infty$. Due to the singularity, the above equation have infinitely many solutions. However, if we impose an additional condition that the solution must be bounded, then it can be shown that the solution is unique.
Theorem 3.1. (Uniqueness of the ODE (5) via a stochastic representation) There is at most one bounded solution to the ODE (5). More precisely, suppose \( a(s) \) solves the ODE (5) and \( \sup_{s \in [0, \infty)} |a(s)| \leq C < \infty \) for some constant \( C > 0 \). Then we must have

\[
a(s) = E_s \left[ \exp \left( -s A_T \right) \right] \quad \text{for any } s \geq 0.
\]  

(6)

Thus a bounded solution, if exists, must be unique.

Proof: In terms of \( S(t) \), we can rewrite \( E_s[\exp(-s A_T)] \) as

\[
E_s[\exp(-s A_T)] = E_s \left[ \int_0^\infty \mu \exp \left( - \int_0^t [\mu + S(u)] du \right) dt \right],
\]

(7)

where the notation \( E_s \) means that the process \( \{S(t)\} \) starts from \( s \), i.e. \( S(0) = s \).

First, by Itô’s formula, we have that

\[
M_t := a(S(t)) \exp \left( - \int_0^t [\mu + S(u)] du \right) + \int_0^t \mu \exp \left( - \int_0^u [\mu + S(u)] du \right) dv
\]

is a local martingale. Indeed, since

\[
da(S(t)) = a'(S(t)) [rS(t) dt + \sigma S(t)dW(t)] + \frac{1}{2} a''(S(t)) \sigma^2 S^2(t) dt
\]

\[
= [(S(t) + \mu)a(S(t)) - \mu] dt + a'(S(t)) \sigma S(t)dW(t),
\]

where the last equality follows from the fact that \( a(s) \) solves the ODE (5), and moreover,

\[
d \exp \left( - \int_0^t [\mu + S(u)] du \right) = \exp \left( - \int_0^t [\mu + S(u)] du \right) \cdot \{-[\mu + S(t)]\} dt,
\]

we obtain by some algebra that

\[
M_t = \exp \left( - \int_0^t [\mu + S(u)] du \right) \cdot a'(S(t)) \sigma S(t)dW(t),
\]

which implies that \( \{M_t\} \) is a local martingale. Actually, \( \{M_t\} \) is a true martingale as \( M_t \) is uniformly bounded, \( \sup_{t \geq 0} |M_t| \leq \sup_{t \geq 0} \left\{ Ce^{-\mu t} + \int_0^t \mu e^{-\mu v} dv \right\} = C + 1 < \infty \), because \( S(u) \geq 0 \). Thus, \( a(s) = a(S(0)) = E_s[M_0] = E_a[M_t] \). Letting \( t \to +\infty \), the first term in \( M_t \) goes to zero almost surely because \( a(\cdot) \) is bounded, and therefore

\[
M_t \to \int_0^\infty \mu \exp \left( - \int_0^u \{\mu + S(u)\} du \right) dv,
\]
almost surely. Accordingly, by the dominated convergence theorem,

$$a(s) = E_s \left[ \lim_{t \to \infty} M_t \right] = E_s \left[ \int_0^\infty \mu \exp \left( - \int_0^s \{ \mu + S(u) \} du \right) dv \right] = E[\exp ( - sA_T \mu )],$$

where the last equality holds due to (7). The theorem is proved. □

Theorem 3.1 implies that if we can find a particular bounded solution to the ODE (5), it must have the stochastic representation in (6). To find such one, consider a difference equation (or a recursion) for a function $H(\nu)$ defined on $(-1, \alpha_1)$

$$h(\nu)H(\nu) = \nu H(\nu - 1) \quad \text{for any } \nu \in (0, \alpha_1), \quad \text{and} \quad H(0) = 1, \quad (8)$$

$$h(\nu) \equiv \mu - G(\nu) = -\frac{\sigma^2}{2} \nu^2 - \left( r - \frac{\sigma^2}{2} \right) \nu + \mu = -\frac{\sigma^2}{2} (\nu - \alpha_1)(\nu - \alpha_2). \quad (9)$$

In general, if the above difference equation (8) has one solution $H_1(\nu)$, then there exist an infinite number of solutions to (8). In fact, any function in the following class

$$\left\{ H(\nu) = \theta(\nu)H_1(\nu) : \text{for some periodic function } \theta(\nu) \text{ s.t. } \theta(\nu) = \theta(\nu - 1) \text{ for any } \nu \in (0, \alpha_1) \right\}$$

also solves (8). This also partly explains why very few people investigated Asian option pricing just based on this recursion. However, we shall show next that the difference equation (8) has a unique solution if we restrict our attention to random variables.

**Theorem 3.2. (A particular bounded solution to the ODE (5))** If there exists an nonnegative random variable $X$ such that $H(\nu) = E[X^\nu]$ satisfies the difference equation (8), then the Laplace transform of $X$, i.e. $E[e^{-sX}]$, solves the nonhomogeneous ODE (5).

**Proof:** Denote the Laplace transform of $X$ by $y(s) = E[e^{-sx}]$, for $s \geq 0$. Note that for any $a \in (0, \min(\alpha_1, 1))$, we have

$$\int_0^{+\infty} s^{-a} e^{-sx} ds = \Gamma(1 - a) X^{a-1} \quad \text{and} \quad \int_0^{+\infty} s^{-a-1} (e^{-sx} - 1) ds = -\frac{\Gamma(1 - a)}{a} X^a,$$

where the second equality holds due to integration by parts. Taking expectations on both sides of the two equations above and applying Fubini’s theorem yield

$$E[X^{a-1}] = \frac{1}{\Gamma(1 - a)} \int_0^{+\infty} s^{-a} y(s) ds \quad \text{and} \quad E[X^a] = -\frac{a}{\Gamma(1 - a)} \int_0^{+\infty} s^{-a-1} (y(s) - 1) ds.$$
Thus, by the difference equation (8), we have
\[
- \frac{ah(a)}{\Gamma(1 - a)} \int_0^\infty s^{-a-1} (y(s) - 1) \, ds = \frac{a}{\Gamma(1 - a)} \int_0^\infty s^{-a} y(s) \, ds,
\]
i.e.
\[
0 = \int_0^\infty s^{-a-1} [sy(s) + h(a)(y(s) - 1)] \, ds,
\]
where \( h(a) \) is given by (9). Setting \( s = e^{-x} \), and \( z(x) = y(s) - 1 \), we have
\[
0 = \int_{-\infty}^{\infty} e^{ax} \{ e^{-x}(z(x) + 1) + h(a)z(x) \} \, dx, \quad \text{for any } a \in (0, \min(\alpha_1, 1)).
\]
For simplicity of notations, rewrite \( h(a) \) as \( h(a) = h_0a^2 + h_1a + h_2 \), with \( h_0 = -\frac{a^2}{2}, h_1 = -r + \frac{a^2}{2}, \) and \( h_2 = \mu \). Note that integration by parts yields
\[
\int_{-\infty}^{\infty} e^{ax} az(x) \, dx = - \int_{-\infty}^{\infty} e^{ax} z'(x) \, dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{ax} a^2 z(x) \, dx = \int_{-\infty}^{\infty} e^{ax} z''(x) \, dx
\]
because \( [z(x)e^{ax}]^{+\infty}_{x=-\infty} = 0 \) and \( [z'(x)e^{ax}]^{\infty}_{x=-\infty} = 0 \). Then for any \( a \in (0, \min(\alpha_1, 1)) \),
\[
0 = \int_{-\infty}^{\infty} e^{ax} \{ e^{-x}(z(x) + 1) + (h_0a^2 + h_1a + h_2)z(x) \} \, dx \\
= \int_{-\infty}^{\infty} e^{ax} \{ e^{-x}(z(x) + 1) + h_0z''(x) - h_1z'(x) + h_2z(x) \} \, dx.
\]
By the uniqueness of the moment generating function, we have an ODE
\[
h_0z''(x) - h_1z'(x) + h_2z(x) + e^{-x}(z(x) + 1) = 0.
\]
Now transferring the ODE for \( z(x) \) back to one for \( y(s) \), with \( s = e^{-x} \) we have \( z(x) = y(s) - 1, z'(x) = -sy'(s), \) and \( z''(x) = sy'(s) + s^2y''(s) \). Then the ODE becomes
\[
h_0s^2y''(s) + (h_1 + h_0)sy'(s) + (h_2 + s)y(s) = h_2.
\]
Substituting \( h_0, h_1, \) and \( h_2 \) into above, we have a nonhomogeneous ODE as follows
\[
\frac{\sigma^2}{2}s^2y''(s) + rsy'(s) - (s + \mu)y(s) = -\mu,
\]
from which the proof is completed. \( \Box \)
It turns out that we can find a particular nonnegative random variable $\chi$ such that $E[\chi^\nu]$ satisfies the difference equation (8). Then according to Theorem 3.2, $E[e^{-s\chi}]$ is a particular bounded solution to the ODE (5). Thus by Theorem 3.1, we can conclude that $E[e^{-s A_T\mu}] = E[e^{-s\chi}]$, which implies that $A_T\mu = d \chi$, with $=d$ meaning equal in distribution.

**Theorem 3.3.** Under the BSM, we have

$$A_T\mu = d \frac{2}{\sigma^2} \frac{Z(1, -\alpha_2)}{Z(\alpha_1)}$$

and therefore

$$E[A_T\mu^\nu] = \left( \frac{2}{\sigma^2} \right)^\nu \frac{\Gamma(\nu + 1) \Gamma(\alpha_1 - \nu) \Gamma(1 - \alpha_2)}{\Gamma(\alpha_1) \Gamma(-\alpha_2 + \nu + 1)},$$

for any $\nu \in (-1, \alpha_1)$.

(10)

Here $Z(a, b)$ denotes a beta random variable with parameters $a$ and $b$, $Z(a)$ a gamma random variable with parameters 1 and $a$, and $\gamma(\cdot)$ denotes the gamma function. Moreover, $Z(1, -\alpha_2)$ and $Z(\alpha_1)$ are independent with $\alpha_1$ and $\alpha_2$ given by (4).

**Proof:** Consider a nonnegative random variable $\chi$ such that $\chi = d \frac{2}{\sigma^2} \frac{Z(1, -\alpha_2)}{Z(\alpha_1)}$. Then some algebra yields

$$E[\chi^\nu] = \left( \frac{2}{\sigma^2} \right)^\nu \frac{\Gamma(\nu + 1) \Gamma(\alpha_1 - \nu) \Gamma(1 - \alpha_2)}{\Gamma(\alpha_1) \Gamma(-\alpha_2 + \nu + 1)},$$

for any $\nu \in (-1, \alpha_1)$,

and furthermore, it can be easily verified that $E[\chi^\nu]$ solves the difference equation (8). By Theorem 3.2, we conclude that $a^*(s) := E[e^{-s\chi}]$ for any $s \geq 0$ is a particular bounded solution to the ODE (5). As a result, it follows from Theorem 3.1 that $A_T\mu = d \chi = d \frac{2}{\sigma^2} \frac{Z(1, -\alpha_2)}{Z(\alpha_1)}$, which gives the distribution of $A_T\mu$. The proof is completed. □

**Remarks:** 1. Note that the recursion (8) and other similar, more general recursions occurred frequently in the literature. Furthermore, it has been proved in various ways that $E[A_T\mu^\nu]$ satisfies this kind of recursion. For example, Carmona, Petit and Yor [9] obtained such recursions for general Levy-type return processes using the associated properties of independent, stationary increments. Dufresne [18] used time reversal and Itô’s formula to derive the recursion (8) for the BSM. In Section 4 of the online supplement, we shall give a new proof that under the HEM, $E[A_T\mu^\nu]$ satisfies a similar recursion only using the Feynman-Kac formula.
2. The result (10) coincides with that in Yor [47] and Matsumoto and Yor [30]. However, compared with the existing proofs involving Bessel processes and Lamperti’s representation, our approach is simpler and more elementary. More importantly, we shall illustrate in Section 4 that our approach is more general, because it can be extended easily to the case of HEM.

3.2 Pricing Formulae for Asian Options and Hedging Parameters under the BSM

Under the BSM, plugging (11) into (1) yields immediately the following Theorem 3.4.

**Theorem 3.4.** Under the BSM, for every \( \mu \) and \( \nu \) such that \( \mu > 0 \) and \( \nu \in (0, \alpha_1 - 1) \), the double-Laplace transform of \( X \mathbb{E}(S^A_t A_t - e^{-k})^+ \) is given by:

\[
L(\mu, \nu) = \frac{X}{\mu \nu (\nu + 1)} \left( \frac{2S_0}{X \sigma^2} \right)^{\nu + 1} \frac{\Gamma(\nu + 2) \Gamma(\alpha_1 - \nu - 1) \Gamma(1 - \alpha_2)}{\Gamma(\alpha_1) \Gamma(\nu - \alpha_2 + 2)}.
\]

Therefore, the Asian option price is equal to:

\[
P(t, k) = e^{-rt} \frac{1}{t} L^{-1}(L(\mu, \nu)) \bigg|_{k=\ln(X/K_t)},
\]

where \( L^{-1} \), a function of \( t \) and \( k \), denotes the Laplace inversion of \( L \). Furthermore, for any maturity \( t \) and strike \( K \), two common greeks delta \( \Delta(P(t, k)) \) and gamma \( \Gamma(P(t, k)) \) can be calculated as follows

\[
\Delta(p(t, k)) = \frac{\partial}{\partial S_0} P(t, k) = e^{-rt} \frac{1}{t} L^{-1} \left( \frac{XS_0^\nu}{\mu} \left( \frac{2}{X \sigma^2} \right)^{\nu + 1} \frac{\Gamma(\nu + 2) \Gamma(\alpha_1 - \nu - 1) \Gamma(1 - \alpha_2)}{\Gamma(\alpha_1) \Gamma(\nu - \alpha_2 + 2)} \right) \bigg|_{k=\ln(X/K_t)}
\]

\[
\Gamma(p(t, k)) = \frac{\partial^2}{\partial^2 S_0} P(t, k) = e^{-rt} \frac{1}{t} L^{-1} \left( \frac{XS_0^{\nu - 1}}{\mu} \left( \frac{2}{X \sigma^2} \right)^{\nu + 1} \frac{\Gamma(\nu + 2) \Gamma(\alpha_1 - \nu - 1) \Gamma(1 - \alpha_2)}{\Gamma(\alpha_1) \Gamma(\nu - \alpha_2 + 2)} \right) \bigg|_{k=\ln(X/K_t)}.
\]

*Proof.* Combining (11) with (1) yields (12). The two greeks can be obtained by interchanging derivatives and integrals based on Theorem A. 12 on pp. 203-204 in Schiff [36]. \( \square \)

Theorem 3.4 requires \( (0, \alpha_1 - 1) \) to be nonempty, i.e. \( \alpha_1 > 1 \), which means some Laplace parameter \( \mu > 0 \) may be disqualified. Nonetheless, a broad range of \( \mu \) meets the requirement. For example, it is sufficient to have \( \mu > r \), because \( \alpha_1 \) solves the equation \( G(x) = \mu \) and \( G(x) - \mu \) is a continuous function. This does not present any difficulty in term of numerical Laplace inversion, as we can choose the Laplace parameter \( \mu \) to meet the requirement.
4 Pricing Asian Options under the HEM

In the HEM, the asset return process \( \{X(t) : t \geq 0\} \) under a risk-neutral measure \( \mathbb{P} \) is given by

\[
X(t) = \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta \right) t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad X(0) = 0,
\]

where \( r \) is the risk-free rate, \( \sigma \) the volatility, \( \zeta := E(e^{Y_1}) - 1 = \sum_{i=1}^{\infty} \frac{p_i \eta_i}{\eta_i - 1} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + 1} - 1 \), \( \{W(t) : t \geq 0\} \) the standard Brownian motion, \( \{N(t) : t \geq 0\} \) a Poisson process with rate \( \lambda \), \( \{Y_i : i \in \mathbb{N}\} \) i.i.d. double-exponentially distributed random variables with the probability density function (pdf)

\[
f_Y(y) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} I_{\{y \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j y} I_{\{y < 0\}}, \tag{16}
\]

with \( p_i > 0, \eta_i > 1, \) for \( i = 1, \ldots, m, \) \( q_j > 0, \eta_j > 0, \) for \( j = 1, \ldots, n, \) and \( \sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1. \)

Due to the jumps, the risk-neutral measure is not unique. Here we assume the risk-neutral measure \( \mathbb{P} \) is chosen within a rational expectations equilibrium setting such that the equilibrium price of an option is given by the expectation under \( \mathbb{P} \) of the discounted option payoff. For details, we refer to Kou [25]. It is worth mentioning that when \( m = n = 0 \) the HEM is reduced to the BSM, and when \( m = n = 1 \) the HEM is reduced to the double exponential jump diffusion model.

The Lévy exponent of \( \{X_t\} \) is given by

\[
G(x) := \frac{E[e^{xX(t)}]}{t} = \frac{1}{2} \sigma^2 x^2 + \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta \right) x + \lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right) \tag{17}
\]

for any \( x \in (-\theta_1, \eta_1) \). Cai [6] showed that for any \( \mu > 0 \), the equation \( G(x) = \mu \) has exactly \( (m+n+2) \) roots \( \beta_{1,\mu}, \ldots, \beta_{m+1,\mu}, \gamma_{1,\mu}, \ldots, \gamma_{n+1,\mu} \) satisfying

\[
-\infty < \gamma_{n+1,\mu} < -\theta_n < \gamma_{n,\mu} < \cdots < -\theta_1 < \gamma_{1,\mu} < 0 < \beta_{1,\mu} < \eta_1 < \cdots < \beta_{m,\mu} < \eta_m < \beta_{m+1,\mu} < \infty. \tag{18}
\]

Additionally, the infinitesimal generator of \( \{S(t) = S(0)e^{X(t)} : t \geq 0\} \) is given by

\[
Lf(s) = \frac{\sigma^2}{2} s^2 f''(s) + (r - \lambda \zeta) s f'(s) + \lambda \int_{-\infty}^{+\infty} [f(se^u) - f(s)] f_Y(u) du, \tag{19}
\]

for any twice continuously differentiable function \( f(\cdot) \).
4.1 Distribution of $A_{T_{\mu}}$ under the HEM

Consider the following nonhomogeneous ordinary integro-differential equation (OIDE)

$$Ly(s) = (s + \mu)y(s) - \mu,$$

where $L$ is given by (19). Similarly as in the BSM, we have the following theorem.

**Theorem 4.1. (Uniqueness of the OIDE (20) via a stochastic representation)** There is at most one bounded solution to the OIDE (20). More precisely, suppose $a(s)$ solves the OIDE (20) and $\sup_{s \in [0, \infty)} |a(s)| \leq C < \infty$ for some constant $C > 0$. Then we must have

$$a(s) = E \left[ \exp \left( -sA_{T_{\mu}} \right) \right] \text{ for any } s \geq 0.$$  \hspace{1cm} (21)

Thus the bounded solution is unique.

**Proof:** See Section 1 in the online supplement. \(\square\)

Since the proofs for Theorem 3.1 and 4.1 only involve Itô’s formula (see Section 1.2 and 1.3 of Øksendal and Sulem [33] or Applebaum [3] for more general Itô formulae), the result holds for more general underlying process $S(t)$ such as exponential Lévy processes and Lévy diffusions.

Next, we look for a particular bounded solution to the OIDE (20), which has the stochastic representation in (21). Consider a difference equation (or a recursion) for a function $H(\nu)$ defined on $(-1, \beta_1)$

$$h(\nu)H(\nu) = \nu H(\nu - 1) \quad \text{for any } \nu \in (0, \beta_1), \quad \text{and} \quad H(0) = 1,$$  \hspace{1cm} (22)

where

$$h(\nu) \equiv \mu - G(\nu) = \mu - \frac{1}{2} \sigma^2 x^2 - (r - \frac{1}{2} \sigma^2 - \lambda \zeta)x + \lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right)$$

$$= \left( \frac{\sigma^2}{2} \right) \frac{\prod_{i=1}^{m+1} (\beta_i - \nu) \prod_{j=1}^{n+1} (-\gamma_j + \nu)}{\prod_{i=1}^{m} (\eta_i - \nu) \prod_{j=1}^{n} (\theta_j + \nu)}.$$  \hspace{1cm} (23)

Here $\beta_1, \ldots, \beta_{m+1}, \gamma_1, \ldots, \gamma_{n+1}$ are actually $\beta_{1,\mu}, \ldots, \beta_{m+1,\mu}, \gamma_{1,\mu}, \ldots, \gamma_{n+1,\mu}$, which are $(m + n + 2)$ roots of the equation $G(x) = \mu$ satisfying the condition (18).
Theorem 4.2. (A particular bounded solution to the OIDE (20)) If there is a nonnegative random variable $X$ such that $H(\nu) = E[X^\nu]$ satisfies the difference equation (22), then the Laplace transform of $X$, i.e. $E[e^{-sX}]$, solves the nonhomogeneous OIDE (20).

Proof: See Section 2 in the online supplement. □

Theorem 4.3. Under the HEM, we have

$$A_{\nu} = \frac{2}{\sigma^2} \frac{Z(1,-\gamma_1) \prod_{j=1}^n Z(\theta_j + 1, -\gamma_j + \theta_j)}{Z(\beta_{m+1}) \prod_{i=1}^m Z(\beta_i, \eta_i - \beta_i)}$$

(24)

and therefore for any $\nu \in (-1, \beta_1)$,

$$E[A_{\nu, t}] = \left( \frac{2}{\sigma^2} \right)^\nu \frac{\Gamma(1 + \nu) \Gamma(1 - \gamma_1)}{\Gamma(1 - \gamma_1 + \nu)} \cdot \prod_{j=1}^n \frac{\Gamma(\theta_j + 1 + \nu) \Gamma(1 - \gamma_j + 1)}{\Gamma(1 - \gamma_j + 1 + \nu) \Gamma(\theta_j + 1)} \cdot \prod_{i=1}^m \frac{\Gamma(\beta_i - \nu) \Gamma(\eta_i)}{\Gamma(\eta_i - \nu) \Gamma(\beta_i)} \cdot \frac{\Gamma(\beta_{m+1} - \nu)}{\Gamma(\beta_{m+1})}.$$  

(25)

Proof: See Section 3 in the online supplement. □

4.2 Pricing Formulae for Asian Options and Hedging Parameters under the HEM

Theorem 4.4. Under the HEM, for every $\mu$ and $\nu$ such that $\mu > 0$ and $\nu \in (0, \beta_1 - 1)$, the double-Laplace transform of $XE(S_0 A_t - e^{-k})^+$ is given by:

$$\mathcal{L}(\mu, \nu) = \frac{X}{\mu \nu (\nu + 1)} \left( \frac{2S_0}{X \sigma^2} \right)^{\nu + 1} \frac{\Gamma(2 + \nu) \Gamma(1 - \gamma_1)}{\Gamma(2 - \gamma_1 + \nu)} \cdot \prod_{j=1}^n \left[ \frac{\Gamma(\theta_j + 2 + \nu) \Gamma(1 - \gamma_j + 1)}{\Gamma(2 - \gamma_j + 1 + \nu) \Gamma(\theta_j + 1)} \right] \cdot \prod_{i=1}^m \left[ \frac{\Gamma(\beta_i - \nu - 1) \Gamma(\eta_i)}{\Gamma(\eta_i - \nu - 1) \Gamma(\beta_i)} \right] \cdot \frac{\Gamma(\beta_{m+1} - \nu - 1)}{\Gamma(\beta_{m+1})}.$$  

(26)

Therefore, the Asian option price is equal to

$$P(t, k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1}(\mathcal{L}(\mu, \nu)) \bigg|_{k=\ln(X/Kt)}.$$  

And two common greeks delta $\Delta(P(t, k))$ and gamma $\Gamma(P(t, k))$ can be calculated as follows

$$\Delta(p(t, k)) = \frac{\partial}{\partial S_0} P(t, k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1} \left( \frac{XS_0^\nu}{\mu \nu} \frac{2}{X \sigma^2} \right)^{\nu + 1} \cdot \frac{\Gamma(2 + \nu) \Gamma(1 - \gamma_1)}{\Gamma(2 - \gamma_1 + \nu)} \cdot \prod_{j=1}^n \left[ \frac{\Gamma(\theta_j + 2 + \nu) \Gamma(1 - \gamma_j + 1)}{\Gamma(2 - \gamma_j + 1 + \nu) \Gamma(\theta_j + 1)} \right] \cdot \prod_{i=1}^m \left[ \frac{\Gamma(\beta_i - \nu - 1) \Gamma(\eta_i)}{\Gamma(\eta_i - \nu - 1) \Gamma(\beta_i)} \right] \cdot \frac{\Gamma(\beta_{m+1} - \nu - 1)}{\Gamma(\beta_{m+1})} \bigg|_{k=\ln(X/Kt)}.$$
\[
\Gamma(p(t, k)) = \frac{\partial^2}{\partial^2 S_0} P(t, k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1} \left( \frac{XS_0^{\nu-1}}{\mu} \left( \frac{2}{X \sigma^2} \right)^{\nu+1} \cdot \frac{\Gamma(2 + \nu) \Gamma(1 - \gamma_1)}{\Gamma(2 - \gamma_1 + \nu)} \right)
\times \prod_{j=1}^{n} \left[ \Gamma(\theta_j + 2 + \nu) \Gamma(1 - \gamma_{j+1}) \right] \cdot \prod_{i=1}^{m} \left[ \Gamma(\beta_i - \nu - 1) \Gamma(\eta_i) \right] \cdot \frac{\Gamma(\beta_{m+1} - \nu - 1)}{\Gamma(\beta_{m+1})} \right]_{k = \ln(X/Kt)}
\]

**Proof.** Combining (25) with (1) yields (26). The hedging parameters can be obtained similarly as in Theorem 3.3. \( \square \)

### 5 Pricing Asian Options via a Two-Sided Euler Inversion Algorithm with a Scaling Factor

In this section, we intend to price Asian options under both the BSM and the HEM by inverting \( \mathcal{L}(\mu, \nu) \) in (12) and (26) numerically. The algorithm used here is proposed in Petrella [34], which, as a generalization of the one-sided Euler inversion algorithm ([1] and [12]), introduces a two-sided Euler inversion with a scaling factor.

The inversion formula in Petrella [34] to get \( f(t, k) \) from its Laplace transform \( \mathcal{L}(\mu, \nu) \) is

\[
f(t, k) = \frac{\exp(A_1/2 + A_2/2)}{4tk} \times \left\{ \mathcal{L} \left( \frac{A_1}{2t}, \frac{A_2}{2k} \right) + 2 \sum_{s=0}^{\infty} (-1)^s Re \left[ -\mathcal{L} \left( \frac{A_1}{2t}, \frac{A_2}{2k} \cdot \frac{i\pi}{k} - \frac{is\pi}{k} \right) \right] \right.
\]

\[
+ 2 \sum_{s=0}^{\infty} (-1)^s Re \left[ \sum_{j=0}^{\infty} (-1)^s L \left( \frac{A_1}{2t} - \frac{i\pi}{t} - \frac{ij\pi}{t}, \frac{A_2}{2k} - \frac{i\pi}{k} - \frac{is\pi}{k} \right) \right] \right.
\]

\[
+ 2 \sum_{j=0}^{\infty} (-1)^j Re \left[ \sum_{s=0}^{\infty} (-1)^s L \left( \frac{A_1}{2t} - \frac{i\pi}{t} - \frac{ij\pi}{t}, \frac{A_2}{2k} + \frac{i\pi}{k} + \frac{is\pi}{k} \right) \right] \}
\]

\[ - e^+_d - e^-_d, \tag{28} \]

where two errors are

\[
e^+_d = \sum_{j_2=1}^{\infty} \sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} f((2j_1 + 1)t, (2j_2 + 1)k) + \sum_{j_1=1}^{\infty} e^{-j_1 A_1} f((2j_1 + 1)t, k), \tag{29} \]

\[
e^-_d = \sum_{j_2=-\infty}^{-1} \sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} f((2j_1 + 1)t, (2j_2 + 1)k), \tag{30} \]

15
and $A_1$ and $A_2$ (to be specified later) are some inversion parameters used to control the errors.

The inversion appears to be accurate, stable, and easy to implement. For example, under the BSM, the prices produced via our algorithm highly agree with benchmarks generated by the other three important methods by Linetsky, Geman-Yor-Shaw, and Vecer, and it is stable even for low volatilities (e.g. $\sigma = 0.05$). The algorithm is easy to implement primarily because the closed-form Laplace transform involves only gamma functions. In addition, a contribution of the current paper is that we arrive at a verifiable discretization error bound in the case of Asian options by adapting the method in Petrella [34]; see Theorem 5.1.

In the computation, we have to calculate alternating series of the form $\sum_{i=0}^{\infty}(-1)^i a_i$ in the expression (28). To accelerate the convergence rate, we adopt the idea of Euler transformation (see [1] and [12]), and approximate $\sum_{i=0}^{\infty}(-1)^i a_i$ by $E(m, n) := \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} 2^{-m} S_{n+k}$, where $S_j := \sum_{i=0}^{j} (-1)^i a_i$. Since there are two transforms, Euler transformation will be used twice. We use $(n_1, m_1)$ and $(n_2, m_2)$ to express parameters involved in Euler transformation for Euler inversion w.r.t. $t$ and $k$, respectively. More precisely, for the first infinite sum and the inner series of both the third and fourth double sum on the RHS of (28), we use Euler transformations with parameters $(n_2, m_2)$; while for the second infinite sum and the outer series of both the third and fourth double sum on the RHS of (28), we use Euler transformation with parameters $(n_1, m_1)$. Suggested by Abate and Whitt [1], we shall set $m_1 = n_1 + 15$, and $m_2 = n_2 + 15$.

In the inversion algorithm there are several parameters to be chosen. (1) $(n_1, n_2)$. The larger $n_1$ and $n_2$ will lead to better accuracy at the cost of computation time. (2) $(A_1, A_2)$. For discretization error control. (3) The scaling factor $X$. One nice feature of the inversion algorithm is its robustness. Under the BSM, this is illustrated in Figure 1, where we can see that the numerical results are insensitive to the choices of $A_1$, $A_2$, and $X$ even for low volatility.

5.1 Pricing Asian Options under the BSM

First, we would like to address the parameters selection under the BSM. According to our calculation, $n_1 = n_2 = 35$ seems to achieve excellent accuracy even for low volatilities (e.g.
\( \sigma = 0.05 \); usually even \( n_1 = 15 \) and \( n_2 = 35 \) can yield good accuracy if the volatility is not low. Moreover, we show in Figure 1 that for broad regions of \( A_1 \in [15, 75] \) and \( A_2 \in [30, 60] \), the numerical results are almost identical. For convenience, we simply select \( A_1 = 28 \), \( A_2 = 40 \) for practical implementation, although one can freely choose other values. Regarding the scaling factor \( X \), Petrella [34] suggested that we can always invert at a fixed point 

\[
k = \min\left(\frac{A_2}{2/(\sigma \sqrt{t})}, A_2/4\right),
\]

no matter what the strike price \( K \) and volatility \( \sigma \) are. Note that with this fixed \( k \), the scaling parameter

\[
X = Kt \cdot e^k = Kt \cdot \exp\left\{ \min\left(\frac{A_2}{2/(\sigma \sqrt{t})}, A_2/4\right) \right\}.
\]

(31)

Figure 1 indicates that within a wide range, the selection of \( X \) other than (31) will lead to almost identical numerical results.

5.1.1 Comparison of Accuracy with Other Methods

To check the accuracy of our double-Laplace inversion algorithm for Asian option pricing, we shall consider seven test cases in Table 1, which are frequently used in the literature such as [19], [28], [40], [15], [45], etc.

<table>
<thead>
<tr>
<th>Case No.</th>
<th>( S_0 )</th>
<th>( K )</th>
<th>( r )</th>
<th>( \sigma )</th>
<th>( t )</th>
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</table>

Table 1: Seven useful test cases in the literature.

A detailed comparison of accuracy between our double-Laplace inversion algorithm with those obtained from seven other existing methods is given in Table 2. From Table 2, we can see easily that our DL prices highly agree with benchmarks of existing methods. Specifically, our DL prices agree with both Geman-Yor-Shaw’s and Linetsky’s results to ten decimal points and with Vecer’s results to six decimal points. Moreover, our DL prices also agree with Zhang’s PDE results to six decimal points. These consistence strongly indicates that our double-Laplace
inversion algorithm are very accurate and can also be used as benchmarks for pricing Asian options.

### Table 2: Comparison of accuracy with other existing methods.

The parameters associated with our double-Laplace inversion method are $n_1 = 3$, $n_2 = 5$, $A_1 = 28$, $A_2 = 40$, and $X$ given by (31). More precisely, DL prices are obtained via our double-Laplace inversion algorithm. Results of Linetsky’s eigenfunction expansion method are cited from Table 3 on p. 866 of [28]. Vecer’s PDE results are from Table A on p. 115 of [45]. The “GYS-Mellin” data based on Shaw’s Mellin transformed-based approximation for Geman and Yor’s single-Laplace inversion method is cited from [40]. Other three columns, including “GY-Shaw”, “Zhang” and “MAE3” that are all cited from the Table on p. 383 of [15], correspond to Geman and Yor’s single-Laplace inversion method along with Shaw’s Mathematica approach [38] (abbreviated GY-Shaw method), Zhang’s PDE method [48], and Dewynne and Shaw’s asymptotic algorithm to the third order [15], respectively. Our numerical DL prices are calculated using Matlab 7.1 with a Pentium M 1.86GHz processor.

Although Shaw’s GYS-Mellin results and Dewynne and Shaw’s MAE3 results seem less accurate than other methods, they are still sufficiently accurate in practice and moreover, these two methods have their own advantages. First, Shaw’s GYS-Mellin method turns out to be very fast. For example, in Case 1 in Table 2 and on a desktop with 2.66 GHz, it takes only 0.047 seconds to produce one result; whereas our pricing method requires 0.563 seconds to match the GYS-Mellin result to five decimal points. Second, these two methods work better than most other methods as the volatility is extremely low; for details, see Section 5.1.2.

The approaches by us, Linetsky, Geman-Yor-Shaw, and Vecer use different methodologies but all provide highly accurate numerical results agreeing with one another. Which method is appropriate to use largely depends on one’s particular needs and environment. For example, Linetsky’s method is easy to program using special functions in Mathematica. Numerical integrations involved in GY-Shaw method can be more conveniently implemented also in Mathematica (it is worth mentioning that the GY-Shaw method can be very easily implemented with no more than fourteen lines of code in Mathematica; see [38]). Vecer’s method is easy to implement if one has a PDE solver such as the one in Mathematica. Our method is perhaps more
Extension of Seven Cases in Table 1 as $\delta > r$ (see Section 6.2 in [15]).

<table>
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<tr>
<th>Case</th>
<th>$S_0$</th>
<th>$K$</th>
<th>$r$</th>
<th>$\delta$</th>
<th>$\sigma$</th>
<th>$t$</th>
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Extension of Seven Cases in Table 1 as $\delta = r$ (see Section 6.3 in [15]).

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<td>0.115188</td>
</tr>
<tr>
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<tr>
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<td>0.05</td>
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<td>0.169238</td>
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<td>0.05</td>
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<tr>
<td>7**</td>
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<td>0.291315</td>
<td>0.291264</td>
<td>0.291315</td>
</tr>
</tbody>
</table>

Table 3: Comparison of accuracy in extended cases when $\delta > r$ or $\delta = r$

suitable if one expects to implement the code directly, such as in C++ or Matlab, without using packages for numerical integrations or PDE solvers. Besides this main point, there are only some minor points. Linetsky’s series formula may converge slowly in the case of low volatility. For example, in Case 1 of Table 2, 400 terms have to be summed to approximate the sum of the series. Linetsky’s series expansion is an analytical solution, while Geman-Yor-Shaw’s, Vecer’s and our methods are numerical solutions. Vecer’s method also works for discrete Asian options. Having these different methods available can offer users more choices, and can also be used to check accuracy and to set up numerical benchmarks for Asian options.

5.1.2 Comparison of Behaviors for Low Volatilities

It is well known that many numerical methods for Asian option pricing do not perform well for low volatilities; see [13, 19, 39]. However, along with Shaw’s elegant Mathematica approach, the revised single-Laplace inversion method, namely the GY-Shaw method (or its variants GYS-Full and CIBess, see [15]), works very well for low volatilities, e.g., $\sigma = 0.05$. Here we would like to conduct cross-comparisons of behaviors for extremely low volatilities between our double-Laplace inversion method and the three methods discussed in [15], GY-Shaw (or its variants
GYS-Full and CIBess), MAE3, and Zhang’s method, for three cases $\delta < r$, $\delta > r$ and $\delta = r$, where $\delta$ denotes the dividend (in [15] the dividend is denoted by $q$). Note that although in our theorems the dividend $\delta = 0$, the theorems can be easily extended to the case of nonzero dividends.

<table>
<thead>
<tr>
<th>Case</th>
<th>$S_0$</th>
<th>$K$</th>
<th>$r$</th>
<th>$\delta$</th>
<th>$\sigma$</th>
<th>$t$</th>
<th>DL Prices</th>
<th>GY-Shaw</th>
<th>MAE3</th>
<th>Zhang</th>
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</thead>
<tbody>
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<td>0.0559860</td>
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<td>0.0339412</td>
<td>0.0339412</td>
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<td>NA</td>
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<th>$K$</th>
<th>$r$</th>
<th>$\delta$</th>
<th>$\sigma$</th>
<th>$t$</th>
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<td>NA</td>
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<td>3.7993x10^{-7}</td>
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<tr>
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<td>1</td>
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<td>NA</td>
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<td>$o(10^{-72})$</td>
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<table>
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<th>$r$</th>
<th>$\delta$</th>
<th>$\sigma$</th>
<th>$t$</th>
<th>DL Prices</th>
<th>CIBess</th>
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<th>Zhang</th>
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</thead>
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<td>0.02</td>
<td>0.1</td>
<td>1</td>
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<td>0.0451431</td>
<td>0.0451431</td>
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<tr>
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<td>0.02</td>
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<td>0.0225755</td>
<td>0.0225755</td>
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</tr>
<tr>
<td>1B**</td>
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</tr>
<tr>
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<td>0.02</td>
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<tr>
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<td>0.02</td>
<td>0.02</td>
<td>0.001</td>
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<td>NA</td>
<td>NA</td>
<td>0.000451537</td>
<td>0.000451537</td>
</tr>
</tbody>
</table>

Table 4: Comparison of accuracy for extremely low volatilities when $\delta > r$, $\delta > r$ or $\delta = r$

First of all, we point out that for the two families of the extended seven cases in Section 6.2 ($\delta > r$) and Section 6.3 ($\delta = r$) in [15], our DL prices, GY-Shaw prices (or its variant CIBess prices) and Zhang’s results agree with one another to six or seven decimal points, hence being more accurate than MAE3. See Table 3. Second, when the volatility is extremely small ($\leq 0.01$) in the three cases of $\delta < r$, $\delta > r$ and $\delta = r$ (the parameter settings are the same as in Section 6.1, 6.2, and 6.3 in [15]), we find from Table 4 that MAE3 and Zhang’s results are still available and highly agree with one another, but our double-Laplace inversion method and GY-Shaw methods do not work. Consequently, we draw the conclusion that our methods and GY-Shaw methods are reliable in the case of normal or reasonably low volatilities ($\sigma \geq 0.05$); we, however, may have to resort to Dewynne and Shaw’s asymptotic method or Zhang’s PDE
method for the extremely low volatilities ($\sigma < 0.05$).

### 5.1.3 Comparison with the Fourier and Laplace Inversion Algorithm

Under the BSM, Fusai [20] gave a closed-form for the Fourier-Laplace transform of Asian option price w.r.t. $k = \ln(\sigma^2Kt/(4S_0))$ and $h = \sigma^2t/4$, respectively. Despite some similarities, there are some key differences between our method and Fusai’s method. (1) Our method performs better for low volatility, e.g., $\sigma = 0.05$ or 0.1. Specifically, for Fusai’s method, a large number of terms are needed to do the Euler inversion to achieve a desired accuracy. In comparison, our algorithm in general requires much less terms in computation, especially for low volatility. See Table 5. This is mainly because we use the latest inversion method with a scaling factor in Petrella [34]. (2) Our method performs better in jump diffusion models. See Section 5.2. (3) Under the BSM, we can derive a theoretical discretization error bound for the double-Laplace inversion. See Section 5.1.5. (4) In terms of the main theoretical difference, note that the recursion used in Fusai’s paper, namely (8), have no unique but infinitely many solutions. We spend considerable efforts to overcome this difficulty. See Section 3.

<table>
<thead>
<tr>
<th></th>
<th>Double-Laplace Inversion Method</th>
<th></th>
<th>Fourier-Laplace Inversion Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 decimal</td>
<td>3 decimal</td>
<td>4 decimal</td>
</tr>
<tr>
<td>$\sigma=0.05$</td>
<td>$(n_1, n_2)=(35,35)$*</td>
<td>(35,35)</td>
<td>(35,35)</td>
</tr>
<tr>
<td>(CPU time)</td>
<td>(3.5 secs)**</td>
<td>(3.5 secs)</td>
<td>(3.5 secs)</td>
</tr>
<tr>
<td>$\sigma=0.1$</td>
<td>$(n_1, n_2)=(15,15)$</td>
<td>(15,35)</td>
<td>(15,35)</td>
</tr>
<tr>
<td>(CPU time)</td>
<td>(1.2 secs)</td>
<td>(2.0 secs)</td>
<td>(2.0 secs)</td>
</tr>
</tbody>
</table>

Table 5: Comparison of the efficiency between our double-Laplace inversion and Fusai’s Fourier-Laplace inversion method. In this table, $(n_1, n_2)=(35,35)$* means that $(n_1, n_2)$ should be set roughly at least (35,35) to achieve 2-decimal accuracy. (3.5 secs)** below $(n_1, n_2)=(35,35)$* means that the corresponding CPU time is 3.5 seconds. The CPU times associated with Fusai’s method are obtained using the code implemented by Matlab 7.1 on an IBM laptop with a Pentium M 1.86GHz processor. We can see that to achieve the same accuracy, our method is more efficient than Fusai’s.
5.1.4 Stability of the Method

To study the stability of the method, we shall also perform some numerical experiments to show the absolute and relative errors of our double-Laplace inversion method against various choices of parameters $A_1$, $A_2$ and $X$. The “true” prices will be obtained by using Monte Carlo simulation with a control variate being $\int_0^t e^{X(s)}ds$, because it is easy to compute $E\left[\int_0^t e^{X(s)}ds\right] = (e^{rt} - 1)/r$, and $\int_0^t e^{X(s)}ds$ has a high degree of correlation with the payoff function. In addition, Richardson extrapolation is also employed to reduce the discretization bias generated when we discretize the sample path to approximate the integral. More precisely, let $M(h)$ be the Monte Carlo estimator without Richardson extrapolation when the discretization step size is set to be $h$. Then we use $(4M(h) - M(2h))/3$ rather than $M(h)$ as the final estimator to achieve the discretization bias reduction. For more details about the technique of control variates and Richardson extrapolation, see Glasserman [22].

Figures 1 shows how the absolute and relative errors change as $A_1$, $A_2$ and $X$ vary in the case of low volatility $\sigma = 0.05$, illustrating that our algorithm is insensitive to the selection of parameters $A_1$, $A_2$ and $X$. For normal volatilities, our method becomes even more stable and associated plots can be obtained on request.

5.1.5 Discretization Error Bounds of Euler Inversion Algorithm under the BSM

The discretization error bound of Euler inversion algorithm was first studied by Abate and Whitt [1], and was extended to a two-sided Laplace inversion case by Petrella [34]. In this subsection, by extending the results in Petrella [34], we shall provide discretization error bounds of the inversion algorithm for our specific case of Asian option pricing under the BSM. The discretization error bounds decay exponentially, therefore leading to a fast convergence.

Recall that what we want to invert is $L(\mu, \nu) = \int_0^\infty \int_{-\infty}^\infty e^{-\mu t} e^{-\nu k} f(t,k)dkdt$, where $f(t,k) = X E(S_t^A A_t - e^{-k})^+$. Then we can prove the following theorem for the error bounds.

**Theorem 5.1.** Suppose $t \in (0, \frac{A_1}{2(\theta_1 + \theta_2)})$ and $k > \frac{A_2}{\theta_2}$, for some constant $\theta_2 > 0$, where $\theta_1 = \ldots$
Figure 1: The stability and accuracy of the algorithm as $A_1$, $A_2$ and $X$ vary in the case of low volatility $\sigma = 0.05$. The default choices for unvarying algorithm parameters are $A_1 = 28$ and $A_2 = 40$ with $X$ being given by (31). The absolute errors and relative errors are reported on the left and right graphs, respectively. For broad regions of $A_1$, $A_2$ and $X$ our algorithm appears to be stable and accurate, all within the 95% confidence intervals. In fact, the relative errors are all smaller than 0.02%.
$1 + \tilde{r} + \sigma^2 / 2 > 0$ and $\tilde{r} = \max(r - \frac{\sigma^2}{2}, 0)$. Then the discretization error bounds $e_d^+$ and $e_d^-$ satisfy

$$e_d^+ \leq \frac{C^+(\theta_1)}{1 - e^{-(A_1 - 2\theta_1 t)}} \left\{ \frac{e^{-A_2}}{1 - e^{-A_2}} + e^{-(A_1 - 2\theta_1 t)} \right\},$$

(32)

$$e_d^- \leq \frac{1}{1 - e^{-(A_1 - 2(\theta_1 + \theta_2) t)}} e^{-(\theta_2 k - A_2)},$$

(33)

with $C^+(\theta_1) := 2S_0e^{\theta_1 t}$ and $C^-(\theta_1, \theta_2) := 2S_0e^{\sigma^2 \theta_2^2 / 2 + (1 + \tilde{r} + \sigma^2) t - 1\theta_2 + \theta_1 t}$.

Proof: See Section 5 in the online supplement. \(\square\)

For example, consider the case where $r = 0.09$, $\sigma = 0.2$ and $t = 1$, and we use $A_1 = 50$, $A_2 = 40$ and $\theta_2 = 20$. Then $\tilde{r} = 0.07$, $\theta_1 = 1.09$, $t = 1 \in \left(0, \frac{A_1}{2\theta_1}\right) \equiv (0, 1.19)$, and $k = 4 \in \left(\frac{A_2}{\theta_2}, +\infty\right) \equiv (2, +\infty)$. Simple algebra yields that $C^+(\theta_1) \approx e^{6.39}$ and $C^-(\theta_1, \theta_2) \approx e^{16.59}$. Plugging them into (32) and (33), we can get discretization error bounds: $e_d^+ \leq 2.53 \times 10^{-15}$ and $e_d^- \leq 6.80 \times 10^{-11}$. Hence, the discretization error for Asian option price is theoretically no more than $6.80 \times 10^{-11} \times e^{-\tilde{r} / t} \approx 6.22 \times 10^{-11}$.

5.2 Pricing Asian Options under the HEM

In this section, we plan to price Asian options numerically under the HEM by inverting the double-Laplace transform (26) via the two-sided Euler inversion algorithm. Without loss of generality, we concentrate on Kou’s model, which along with the BSM are the most important special cases of the HEM. Tables 6 and 7 give the numerical results of Asian option prices using double-Laplace transform in the cases of $\lambda = 3$ and $\lambda = 5$. The double-Laplace inversion method seem to be accurate under the DEM.

Here we point out that the Fourier-Laplace inversion method seems unstable in the case of low volatilities under the DEM. Specifically, the Fourier-Laplace inversion method is sensitive to parameters. To illustrate the sensitivity, we shall fix $A_f = 40$ and let $A_l$ change from 15 to 38. The right panel of Figure 2 illustrates how the difference between the numerical result and the true value changes as $A_l$ varies in the case of $\sigma = 0.1$ and $K = 100$. In comparison with the double-Laplace inversion on the left panel of Figure 2, FL prices seem unstable. Figure 2
### Table 6: Numerical results of Asian option prices under DEM with \( \lambda = 3 \). Other parameters of the model are set as: \( S_0 = 100, r = 0.09, t = 1.0, p_1 = 0.6, q_1 = 0.4, \) and \( \eta_1 = \theta_1 = 25 \). Parameters of the algorithm are set as: \( n_1 = 35, n_2 = 55, A_1 = 38.9, A_2 = 40; \) DL prices are obtained by double-Laplace inversion; MC price are Monte Carlo simulation estimates obtained by simulating 1,000,000 paths and setting the discretization step size to be 0.0001; Std Err is the standard error of the MC price; Abs Err and Rel Err are absolute and relative errors, respectively.

<table>
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<th>( \sigma )</th>
<th>( K )</th>
<th>DL Prices</th>
<th>MC Prices</th>
<th>Std Err</th>
<th>Abs Err</th>
<th>Rel Err</th>
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<td>13.42054</td>
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<tr>
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<td>95</td>
<td>8.98812</td>
<td>8.98730</td>
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<td>0.00082</td>
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<tr>
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<td>100</td>
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<td>4.95681</td>
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<td>0.00006</td>
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seems to indicate that, with jumps, our double-Laplace inversion method works more stable than Fourier-Laplace method especially in the case of low volatility.

**Acknowledgment**

We would like to thank Mark Broadie, Gianluca Fusai, Guillermo Gallego, Paul Glasserman, Vadim Linetsky, Giovanni Petrella, Jan Vecer, and Ward Whitt for their helpful comments.
Table 7: Numerical results of Asian option prices under DEM with $\lambda = 5$. Other parameters of the model are set as: $S_0 = 100$, $r = 0.09$, $t = 1.0$, $p = 0.6$, $q = 0.4$, $\eta_1 = \eta_2 = 25$. Parameters of the algorithms are $n_1 = 35$, $n_2 = 55$, $A_1 = 35.9$, $A_2 = 40$. DL prices are obtained by the double Laplace inversion; MC prices are Monte Carlo simulation estimates using 1 million paths and the discretization step sizes to be 0.0001.

### References


26
Figure 2: Comparison of the stability and accuracy between the double-Laplace inversion and the Fourier-Laplace inversion method in the case of low volatility $\sigma = 0.1$ under DEM, where other parameters are $K = 100$, $S_0 = 100$, $t = 1$, $\lambda = 3$, $r = 0.09$, $p_1 = 0.6$, $q_1 = 0.4$, and $\eta_1 = \theta_1 = 25$. The absolute errors of LL prices (obtained by the double-Laplace inversion method) and FL prices (obtained by the Fourier-Laplace inversion method) are reported on the left and right graphs, respectively. Other parameters for the left graph are $n_1 = 35$, $n_2 = 55$, $A_2 = 40$ and $X = 5460$; while other parameters for the right graph are $n_1 = 35$, $n_f = 135$ and $A_f = 40$. We can see that LL prices are quite accurate and stable when $A_1$ varies between $[22.2,38]$, but FL prices are so desultory that we cannot decide which $A_l$ we should choose.


1 Proof of Theorem 4.1

Proof: For the HEM, the argument is similar as that for the BSM in Theorem 3.1 except that we need to show the process \( \{M(t)\} \) is still a local martingale in the jump diffusion case. Indeed, by Ito’s formula for jump diffusions, we have

\[
\begin{align*}
da(S(t)) &= a'(S(t-))dS^c(t) + \frac{1}{2} a''(S(t-))d\langle S^c, S^c \rangle(t) + d \sum_{0 < u \leq t} [a(S(u)) - a(S(u-))] \\
&= a'(S(t-)) [(r - \lambda \zeta)S(t-)dt + \sigma S(t-)dW(t)] + \frac{1}{2} a''(S(t-)) \sigma^2 S^2(t-)dt \\
&\quad + d \sum_{0 < u \leq t} [a(S(u)) - a(S(u-))] \\
&= \left[ (r - \lambda \zeta)S(t-)a'(S(t-)) + \frac{1}{2} \sigma^2 S^2(t-)a''(S(t-)) \right] dt + \sigma S(t-)a'(S(t-))dW(t) \\
&\quad + d \sum_{0 < u \leq t} [a(S(u)) - a(S(u-))] \\
&= [(S(t-) + \mu)a(S(t-)) - \mu]dt + \sigma S(t-)a'(S(t-))dW(t) \\
&\quad + d \sum_{0 < u \leq t} [a(S(u)) - a(S(u-))] - \lambda \int_{-\infty}^{+\infty} [a(S(t)e^y) - a(S(t))] f_Y(y)dy dt,
\end{align*}
\]

where the last equality follows from the fact that \( a(s) \) solves the OIDE (20). Note that

\[
\begin{align*}
dM(t) &= \exp \left( - \int_0^t \{\mu + S(u)\}du \right) \cdot da(S(t)) + a(S(t-)) \cdot d\exp \left( - \int_0^t \{\mu + S(u)\}du \right) \\
&\quad + d\left\langle a(S(t), \exp \left( - \int_0^t \{\mu + S(u)\}du \right) \right\rangle + \mu \exp \left( - \int_0^t \{\mu + S(u)\}du \right) dt.
\end{align*}
\]
Plugging $d(a(S(t)))$ into $dM(t)$ yields

\[
dM(t) = \exp\left( -\int_0^t \{\mu + S(u)\} du \right) \left\{ \left[ (S(t-)+\mu)a(S(t-)) - \mu \right] dt + \sigma S(t-)a'(S(t-))dW(t) \right\} \\
+ \exp\left( -\int_0^t \{\mu + S(u)\} du \right) \left\{ \left\{ -\{\mu + S(t-)\} \right\} a(S(t-)) dt + \mu \exp\left( -\int_0^t \{\mu + S(u)\} du \right) dt \\
+ \exp\left( -\int_0^t \{\mu + S(u)\} du \right) d \sum_{0<u\leq t} [a(S(u)) - a(S(u-))] \\
- \lambda \exp\left( -\int_0^t \{\mu + S(u)\} du \right) \int_{-\infty}^{+\infty} [a(S(t-)e^y) - a(S(t-))] f_Y(y) dy dt \\
= \exp\left( -\int_0^t \{\mu + S(u)\} du \right) a'(S(t-)) \sigma S(t-) dW(t) \\
+ \exp\left( -\int_0^t \{\mu + S(u)\} du \right) \left\{ \left\{ -\{\mu + S(t-)\} \right\} a(S(t-)) dt + \mu \right\} \exp\left( -\int_0^t \{\mu + S(u)\} du \right) dt \\
- \lambda \exp\left( -\int_0^t \{\mu + S(u)\} du \right) \int_{-\infty}^{+\infty} [a(S(t-)e^y) - a(S(t-))] f_Y(y) dy dt,
\]

which is a local martingale. Then the same proof as Theorem 3.1 applies. □

2 Proof of Theorem 4.2

Proof: Similar algebra as in Theorem 3.2 yields that

\[
0 = \int_{-\infty}^{+\infty} e^{ax} \left\{ e^{-x}(z(x) + 1) + h(a)z(x) \right\} dx, \quad \text{for any } a \in (0, \min(\beta_1, 1)),
\]

where $\beta_1$ is the smallest positive root of $G(x) = \mu$, and $z(x) = y(s) - 1$. Plugging $h(a)$ in (23) into (34), we have that for all $a \in (0, \min(\beta_1, 1))$,

\[
\int_{-\infty}^{+\infty} e^{ax} \left\{ e^{-x}(z(x) + 1) \\
+ \left[ \mu - \frac{1}{2}\sigma^2a^2 - \left( r - \frac{\sigma^2}{2} - \lambda \zeta \right) a - \lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i - a}{\eta_i - a} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + a} - 1 \right) \right] z(x) \right\} dx = 0
\]
i.e.
\[
\int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x} (z(x) + 1) + \frac{1}{2} \sigma^2 a^2 - \left( r - \frac{\sigma^2}{2} - \lambda \zeta \right) a + \mu \right\} z(x) \right\} dx
\]
\[
+ \int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - a} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + a} - 1 \right) z(x) \right\} dx = 0
\]

On the one hand, using the same technique as in the proof of Theorem 3.2, we can show that the first integral on the left hand side of the above is equal to
\[
\int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x} (z(x) + 1) - \frac{1}{2} \sigma^2 z''(x) + \left( r - \frac{\sigma^2}{2} - \lambda \zeta \right) z'(x) + \mu z(x) \right\} dx.
\]

On the other hand, we claim that the second integral is equal to
\[
\int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \int_{-\infty}^{\infty} [z(x - u) - z(x)] f_Y(u) du \right\} dx.
\]

Indeed, this is because

\[
\int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \int_{-\infty}^{\infty} [z(x - u) - z(x)] f_Y(u) du \right\} dx
\]
\[
= \int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \int_{-\infty}^{\infty} z(x - u) f_Y(u) du \right\} dx + \int_{-\infty}^{\infty} e^{ax} \lambda z(x) dx
\]
\[
= -\lambda \int_{-\infty}^{\infty} f_Y(u) \left\{ \int_{-\infty}^{\infty} e^{ax} z(x - u) dx \right\} du + \int_{-\infty}^{\infty} e^{ax} \lambda z(x) dx
\]
\[
= -\lambda \int_{-\infty}^{\infty} f_Y(u) \left\{ \int_{-\infty}^{\infty} e^{a(u+\pi)} z(\pi) d\pi \right\} du + \int_{-\infty}^{\infty} e^{ax} \lambda z(x) dx
\]
\[
= \int_{-\infty}^{\infty} e^{ax} \lambda z(\pi) \left\{ -\lambda \int_{-\infty}^{\infty} e^{au} f_Y(u) du \right\} d\pi + \int_{-\infty}^{\infty} e^{ax} \lambda z(x) dx
\]
\[
= \int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - a} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + a} - 1 \right) z(x) \right\} dx,
\]

where the third equality is via change of variable \( x - u = \pi \), and the fifth equality holds because \( 0 < a < \beta_1 < \eta_1 \).

Thus we have
\[
\int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x} (z(x) + 1) - \frac{1}{2} \sigma^2 z''(x) + \left( r - \frac{\sigma^2}{2} - \lambda \zeta \right) z'(x) + \mu z(x)
\]
\[
- \lambda \int_{-\infty}^{\infty} [z(x - u) - z(x)] f_Y(u) du \right\} dx = 0, \quad \text{for all} \ a \in (0, \min(\beta_1, 1)).
\]
By the uniqueness of moment generating function we have an OIDE as follows

\[ e^{-x}(z(x)+1) - \frac{1}{2}\sigma^2 z''(x) + \left( r - \frac{\sigma^2}{2} - \lambda \right) z'(x) + \mu z(x) - \lambda \int_{-\infty}^{\infty} [z(x-u) - z(x)] f_Y(u) du = 0. \]

Now transferring the OIDE back to \( y(s) \), with \( s = e^{-x} \) we have

\[ z(x) = y(s) - 1, \quad z'(x) = -sy'(s), \quad z''(x) = sy'(s) + s^2y''(s), \]

\[ z(x-u) = z(-\log s - u) = z(-\log(se^u)) = y(se^u) - 1 \]

and the OIDE becomes

\[ -\frac{\sigma^2}{2}s^2y''(s) - (r - \lambda \zeta) sy'(s) + (s + \mu)y(s) - \lambda \int_{-\infty}^{\infty} [y(se^u) - y(s)] f_Y(u) du = \mu, \]

i.e. \( Ly(s) = (s + \mu)y(s) - \mu \), from which the proof is completed. \( \Box \)

### 3 Proof of Theorem 4.3

**Proof:** Consider a random variable \( \chi \) that satisfies

\[ \chi \overset{d}{=} \frac{2}{\sigma^2} \frac{Z(1, -\gamma_1) \prod_{j=1}^{n} Z(\theta_j + 1, -\gamma_{j+1} - \theta_j)}{Z(\beta_{m+1}) \prod_{i=1}^{m} Z(\beta_i, \eta_i - \beta_i)}, \tag{35} \]

where all the gamma and beta random variables on the RHS are independent. Then some algebra yields

\[
E[\chi^\nu] = \left( \frac{2}{\sigma^2} \right)^\nu \frac{\Gamma(1 + \nu)\Gamma(1 - \gamma_1)}{\Gamma(1 - \gamma_1 + \nu)} \cdot \prod_{j=1}^{n} \left[ \frac{\Gamma(\theta_j + 1 + \nu)\Gamma(1 - \gamma_{j+1} + \nu)}{\Gamma(1 - \gamma_{j+1} + \nu)\Gamma(\theta_j + 1 + \nu)} \right] \cdot \prod_{i=1}^{m} \left[ \frac{\Gamma(\beta_i - \nu)\Gamma(\eta_i)}{\Gamma(\eta_i - \nu)\Gamma(\beta_i)} \right] \cdot \frac{\Gamma(\beta_{m+1} - \nu)}{\Gamma(\beta_{m+1})},
\]

and moreover, we can verify that \( E[\chi^\nu] \) solves the difference equation (22). By Theorem 4.2, we can draw the conclusion that

\[ a^*(s) := Ee^{-s\chi} \text{ for any } s \geq 0 \]

is a particular bounded solution to the OIDE (20). As a result, according to Theorem 4.1, the proof is completed. \( \Box \)
4 A Proposition on the Recursion (22) under the HEM

In this section, we shall give an alternative proof that under the HEM, $E[A_{T\mu}^\nu]$ solves the recursion (22) using the Feynman-Kac formula.

**Proposition 4.1.** Under the HEM, $E[A_{T\mu}^\nu]$ satisfies the recursion (22), i.e.,

$$h(\nu)E[A_{T\mu}^\nu] = \nu E[A_{T\mu}^{\nu-1}]$$

for any $\nu \in (0, \beta_1)$, where $h(\nu)$ is given by (23).

**Proof:** Under the HEM, define $Y(t) = \int_0^t S(u)du$, where $S(u)$ represents the underlying stock price at the time $u$. Consider a function $f^*(t, x, y)$ defined as follows:

$$f^*(t, x, y) := E(Y_T^\nu \mid S(t) = x, Y(t) = y), \quad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R},$$

where $T$ is any fixed positive real number and $\nu \in (-1, \beta_1)$ is a constant. Note that $\{(S(t), Y(t)) : t \geq 0\}$ is a two-dimensional Markov process under HEM so that $f^*(t, S(t), Y(t)) = E(Y_T^\nu \mid \mathcal{F}_t)$.

Accordingly, the multivariate Feynman-Kac theorem implies that $f^*(t, x, y)$ solves the following PIDE:

$$f_t^*(t, x, y) + (r - \lambda \zeta)x f_x^*(t, x, y) + x f_y^*(t, x, y) + \frac{\sigma^2}{2} x^2 f_{xx}^*(t, x, y)$$

$$+ \lambda \int_{-\infty}^{+\infty} [f^*(t, xe^z, y) - f^*(t, x, y)] f_Y(z)dz = 0, \quad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}.$$

On the other hand, by the Markovian property, we can rewrite $f^*(t, x, y)$ as follows.

$$f^*(t, x, y) = E\left[\left(y + x \int_0^{T-t} e^{(r-\lambda \zeta - \frac{\sigma^2}{2})u + \sigma W(u) + \sum_{i=1}^N(u) \right)Y_i du \right]^\nu] = E[(y + x A_{T-t})^\nu].$$

Introduce a new function $\hat{f}^*(t, x, y)$ defined as:

$$\hat{f}^*(t, x, y) := f^*(T - t, x, y) = E[(y + x A_t)^\nu], \quad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R},$$

which satisfies the following PIDE:

$$-\hat{f}_t^*(t, x, y) + (r - \lambda \zeta)x \hat{f}_x^*(t, x, y) + x \hat{f}_y^*(t, x, y) + \frac{\sigma^2}{2} x^2 \hat{f}_{xx}^*(t, x, y)$$

$$+ \lambda \int_{-\infty}^{+\infty} [\hat{f}^*(t, xe^z, y) - \hat{f}^*(t, x, y)] f_Y(z)dz = 0, \quad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}.$$
with the initial value

\[ \hat{f}^*(0, x, y) = y^\nu. \]

It is worth noting that the “T” in the PIDE (37) can be any positive real number.

Next, for any \( \nu \in (0, \beta_1) \), define \( h^*(x, y) = E \left[ (y + xA_{T_\mu})^\nu \right] = \int_0^\infty \mu e^{-\mu t} \hat{f}^*(t, x, y) dt, x \in \mathbb{R}^+, y \in \mathbb{R} \). Taking the Laplace transform on the variable \( t \) of both sides of the PIDE (37) and then multiplying both sides by \( \mu \), we can show that \( h^*(x, y) \) solves a PIDE as follows.

\[
\mu y^\nu - \mu h^*(x, y) + (r - \lambda \zeta) x h^*_x(x, y) + x h^*_y(x, y) + \frac{\sigma^2}{2} x^2 h^*_{xx}(x, y)
+ \lambda \int_{-\infty}^{+\infty} [h^*(xe^z, y) - h^*(x, y)] f_Y^*(z) dz = 0, \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}.
\]

Now interchanging the derivatives and integrals by using Theorem A. 12 on the pp. 203-204 in Schiff [36], we have

\[
\begin{align*}
    x h^*_x(x, y) &= \nu E \left[ (y + xA_{T_\mu})^\nu \right] - \nu y E \left[ (y + xA_{T_\mu})^{\nu-1} \right], \\
    x h^*_y(x, y) &= \nu x E \left[ (y + xA_{T_\mu})^{\nu-1} \right], \\
    x^2 h^*_{xx}(x, y) &= \nu (\nu - 1) E \left[ (y + xA_{T_\mu})^\nu \right] + \nu (\nu - 1) y^2 E \left[ (y + xA_{T_\mu})^{\nu-2} \right] \\
&- \nu (\nu - 1) 2y E \left[ (y + xA_{T_\mu})^{\nu-1} \right].
\end{align*}
\]

Consequently, we obtain

\[
\begin{align*}
    \mu y^\nu + \left( -\mu + (r - \lambda \zeta) \nu + \frac{\sigma^2}{2} \nu (\nu - 1) \right) E \left[ (y + xA_{T_\mu})^\nu \right] \\
+ \nu \left( x - (r - \lambda \zeta) y - \sigma^2 (\nu - 1) y \right) E \left[ (y + xA_{T_\mu})^{\nu-1} \right] + \frac{\sigma^2}{2} \nu (\nu - 1) y^2 E \left[ (y + xA_{T_\mu})^{\nu-2} \right] \\
+ \lambda \int_{-\infty}^{+\infty} \left\{ E \left[ (y + xe^zA_{T_\mu})^\nu \right] - E \left[ (y + xA_{T_\mu})^\nu \right] \right\} f_Y(z) dz = 0.
\end{align*}
\]

In the special case \( x = 1 \) and \( y = 0 \), since

\[
\int_{-\infty}^{+\infty} \left[ E \left( e^{z\nu A_{T_\mu}} \right) - E \left( A_{T_\mu}^\nu \right) \right] f_Y(z) dz = E \left( A_{T_\mu}^\nu \right) \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - \nu} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + \nu} - 1 \right),
\]

we obtain that for any \( \nu(0, \beta_1) \),

\[
\left[ \frac{\sigma^2}{2} \nu^2 + (r - \lambda \zeta - \frac{\sigma^2}{2}) \nu + \lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - \nu} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + \nu} - 1 \right) - \mu \right] E \left( A_{T_\mu}^\nu \right) = -\nu E \left( A_{T_\mu}^{\nu-1} \right),
\]

which is exactly (36) and (22). The proof is completed. \( \square \)

A-6
5 Proof of Theorem 5.1

Proof. First, since the scaling factor $X > S_0$, we have that
\[ f(t, k) = X E \left( \frac{S_0}{X} A_t - e^{-k} \right)^+ \leq X E \left( \frac{S_0}{X} A_t - \frac{S_0}{X} e^{-k} \right)^+ = S_0 E (A_t - e^{-k})^+, \]
where $k = \log(\frac{X}{A_t})$. On the other hand, we can bound $A_t$ as follows
\[ A_t = \int_0^t e^{(r - \frac{\sigma^2}{2})s + \sigma W(s)} ds \leq \int_0^t \exp \left( \tilde{r} t + \sigma_{\max{0 \leq s \leq t}} W(s) \right) ds = t \exp \left( \tilde{r} t + \sigma_{\max{0 \leq s \leq t}} W(s) \right), \]
where $\tilde{r} := \max(r - \frac{\sigma^2}{2}, 0)$. Since $\max_{0 \leq s \leq t} W(s) = |W(t)|$, it follows that
\[
\begin{align*}
f(t, k) &\leq S_0 E \left[ t \exp \left\{ \tilde{r} t + \sigma |W(t)| \right\} - e^{-k} \right]^+ \\
&= S_0 E \left[ \left( t \exp \left\{ \tilde{r} t + \sigma W(t) \right\} - e^{-k} \right) I_{\{W(t) \geq 0, t \exp(\tilde{r} t + \sigma W(t)) > e^{-k}\}} \right] \\
&\quad + S_0 E \left[ \left( t \exp \left\{ \tilde{r} t - \sigma W(t) \right\} - e^{-k} \right) I_{\{W(t) < 0, t \exp(\tilde{r} t - \sigma W(t)) > e^{-k}\}} \right] \\
&\leq S_0 E \left[ \left( t \exp \left\{ \tilde{r} t + \sigma W(t) \right\} - e^{-k} \right) I_{\{t \exp(\tilde{r} t + \sigma W(t)) > e^{-k}\}} \right] \\
&\quad + S_0 E \left[ \left( t \exp \left\{ \tilde{r} t - \sigma W(t) \right\} - e^{-k} \right) I_{\{t \exp(\tilde{r} t - \sigma W(t)) > e^{-k}\}} \right] \\
&= 2S_0 e^{(1 + \tilde{r} + \sigma^2/2)t} \tilde{P} \{ Y_t > -k - \log(t) \},
\end{align*}
\]
via the symmetric property of standard Brownian motion.

Next, introduce a new measure $\tilde{P}$ such that $\frac{d\tilde{P}}{d\tilde{P}^*} = e^{Y_t - (\tilde{r} + \sigma^2/2)t}$, where $Y_t := \tilde{r} t + \sigma W(t)$. Then the change of measure leads to
\[
\begin{align*}
f(t, k) &\leq 2S_0 E \left[ \exp \{ \tilde{r} t + \sigma W(t) \} I_{\{t \exp(\tilde{r} t + \sigma W(t)) > e^{-k}\}} \right] \times e^{-Y_t + (\tilde{r} + \sigma^2/2)t} \\
&= 2S_0 e^{(\tilde{r} + \sigma^2/2)t} \tilde{P} \{ t \exp(Y_t) > e^{-k} \} \\
&\leq 2S_0 e^{(1 + \tilde{r} + \sigma^2/2)t} \tilde{P} \{ Y_t > -k - \log(t) \} \\
&= 2S_0 e^{\theta_1 t} \tilde{P} \{ Y_t > -k - \log(t) \},
\end{align*}
\]
where $\theta_1 := 1 + \tilde{r} + \sigma^2/2 > 0$ and the last inequality holds because $t < e^t$ for any $t > 0$.

Therefore, when $j_1 \geq 0$ and $j_2 \geq 0$, we have
\[
\begin{align*}
f((2j_1 + 1)t, (2j_2 + 1)k) &\leq 2S_0 e^{\theta_1 (2j_1 + 1)t} \tilde{P} \{ Y_t > -(2j_2 + 1)k - \log((2j_1 + 1)t) \} \\
&\leq 2S_0 e^{\theta_1 t} e^{2\theta_1 j_1 t} = C^+(\theta_1) e^{2\theta_1 j_1 t},
\end{align*}
\]
where $C^+(\theta_1) := 2S_0 e^{\theta_1 t}$. On the other hand when $j_1 \geq 0$ and $j_2 \leq -1$, we have that for any $\theta_2 > 0$,
\[
 f((2j_1 + 1)t, (2j_2 + 1)k) \leq 2S_0 e^{\theta_1(2j_1 + 1)t} \tilde{P}\{Y_t > -(2j_2 + 1)k - \log((2j_1 + 1)t)\} \\
 \leq 2S_0 e^{\theta_1(2j_1 + 1)t} \tilde{P}\{Y_t > -j_2 k - \log((2j_1 + 1)t)\} \\
 \leq 2S_0 e^{\theta_1(2j_1 + 1)t} \tilde{E}\left(e^{\theta_2 Y_t}\right) e^{\theta_2 j_2 k + \theta_2 \log((2j_1 + 1)t)},
\]
where the second inequality holds as $j_2 \leq -1$ and the third inequality comes from Markov’s inequality. Since $x + 1 \leq e^x$ for any $x > -1$, we obtain that $e^{\theta_2 \log((2j_1 + 1)t)} \leq e^{\theta_2[(2j_1 + 1)t - 1]}$ and
\[
 f((2j_1 + 1)t, (2j_2 + 1)k) \leq 2S_0 e^{\theta_1(2j_1 + 1)t} \tilde{E}\left(e^{\theta_2 Y_t}\right) e^{\theta_2 j_2 k + \theta_2 [(2j_1 + 1)t - 1]} \\
 = 2S_0 e^{(\theta_1 + \theta_2)t - \theta_2} \tilde{E}\left(e^{\theta_2 Y_t}\right) e^{2(\theta_1 + \theta_2)j_1 t} e^{\theta_2 j_2 k} \\
 = C^-(\theta_1, \theta_2) e^{\theta_2 j_2 k + 2(\theta_1 + \theta_2)j_1 t},
\]
where
\[
 C^-(\theta_1, \theta_2) := 2S_0 e^{(\theta_1 + \theta_2)t - \theta_2} \tilde{E}\left(e^{\theta_2 Y_t}\right) = 2S_0 e^{(\theta_1 + \theta_2)t - \theta_2} \tilde{E}\left(e^{(\theta_1 + 1)Y_t - (\bar{r} + \sigma^2/2)t}\right).
\]
Recall that $Y_t = \bar{r} t + \sigma W(t)$. Simple algebra yields
\[
 C^-(\theta_1, \theta_2) = 2S_0 e^{t \sigma^2 \theta_2^2 / 2 + [(1 + \bar{r} + \sigma^2)t - 1] \theta_2 + \theta_1 t}.
\]
If we have $t \in \left(0, \frac{A_1}{\theta_2^2}\right)$, according to the definition of $e_d^+$ and the bound of function $f((2j_1 + 1)t, (2j_2 + 1)k)$ obtained above, we can get
\[
 e_d^+ \leq \sum_{j_2=1}^{\infty} \sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} C^+(\theta_1) e^{\theta_1 j_1 t} + \sum_{j_1=1}^{\infty} e^{-j_1 A_1} C^+(\theta_1) e^{\theta_1 j_1 t} \\
 = C^+(\theta_1) \sum_{j_2=1}^{\infty} \sum_{j_1=0}^{\infty} e^{-(A_1 - 2\theta_1 t)j_1 - j_2 A_2} + C^+(\theta_1) \sum_{j_1=1}^{\infty} e^{-(A_1 - 2\theta_1 t)j_1} \\
 = C^+(\theta_1) \frac{e^{-A_2}}{1 - e^{-A_2}} \frac{1}{1 - e^{-(A_1 - 2\theta_1 t)}} + C^+(\theta_1) \frac{e^{-(A_1 - 2\theta_1 t)}}{1 - e^{-(A_1 - 2\theta_1 t)}} \\
 = \frac{C^+(\theta_1)}{1 - e^{-(A_1 - 2\theta_1 t)}} \left\{ \frac{e^{-A_2}}{1 - e^{-A_2}} + e^{-(A_1 - 2\theta_1 t)} \right\},
\]
which is exactly (32).
For $e_d^-$ we have for any $t \in \left(0, \frac{A_1}{2(\theta_1+\theta_2)} \right)$ and $k > \frac{A_2}{\theta_2}$,

$$e_d^- \leq \sum_{j_2=-\infty}^{-1} \sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} C^- (\theta_1, \theta_2) e^{\theta_2 j_2 k + 2(\theta_1 + \theta_2) j_1 t}$$

$$= C^- (\theta_1, \theta_2) \sum_{j_1=0}^{\infty} e^{-(A_1 - 2(\theta_1 + \theta_2)t) j_1} \sum_{j_2=-\infty}^{-1} e^{j_2 (\theta_2 k - A_2)}$$

$$= C^- (\theta_1, \theta_2) \frac{1}{1 - e^{-(A_1 - 2(\theta_1 + \theta_2)t)}} \frac{e^{-(\theta_2 k - A_2)}}{1 - e^{-(\theta_2 k - A_2)}} ,$$

from which (33) is proved. \(\square\)