Default intensities implied by CDO spreads: inversion formula and model calibration

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Bottom-up models: Model individual default rates + “default correlation” structure.

- Static (copula) models. Li (2001).
- Multi-name structural models.
**Credit portfolio models**

**Bottom-up models**: Model individual default rates + “default correlation” structure.
- Static (copula) models. Li (2001).
- Multi-name structural models.

**Top-down models**: Model loss process \( (L_t) \) of the portfolio as an increasing jump process by specifying its intensity \( (\lambda_t) \).
- Local intensity model: \( \lambda_t = F(t, L_t) \). Cont and Minca (2008), Herbertsson (2008), Laurent et al (2007).
- Two factor spread/default model: \( \lambda_t = F(t, L_t, X_t) \). Arnsdorff and Halperin (2008), Lopatin and Misirpashaev (2007).
Motivation

- Although dynamic models are more realistic, they are typically more difficult to estimate. The main obstacle in their implementation has been the lack of stable calibration methods.
- Common practice to calibrate dynamic models: Black-box optimization applied to non-convex least squares minimization.
- Problem: Convergence and stability are not guaranteed.
Motivation

- Although dynamic models are more realistic, they are typically more difficult to estimate. The main obstacle in their implementation has been the lack of stable calibration methods.
- Common practice to calibrate dynamic models: Black-box optimization applied to non-convex least squares minimization.
- Problem: Convergence and stability are not guaranteed.
- We develop a simple method to recover the portfolio default intensity based on an **analytical inversion formula** and **quadratic programming** and compare it with alternative calibration methods: parametric method by Herbertsson (2008) and entropy minimization method by Cont and Minca (2008).
- Comparisons reveal a large amount of model uncertainty in pricing and hedging.
Figure 1: Application of the inversion formula to recover the local intensity function.
Roadmap

Credit portfolio loss models

CDO market Data

Simulation

Quadratic programming

Expected tranche notionals

Inversion formula

Local intensity function

Model comparison

Hedging CDO tranches

Pricing exotic financial products
Local intensity function and Markovian projection

- An equally weighted credit portfolio consisting of $n$ names.
- $N_t$: number of defaults by time $t$.
- $\delta$: loss given default, assumed to be constant.
- $L_t = \delta N_t$: credit portfolio loss at time $t$.
- Assumption: $(N_t)$ admits an intensity ($\lambda_t$).
- Interest rates are independent from default times.

**Definition 1**

Consider a loss process satisfying the above setting with

\[ \forall t \in (0, T^*], \quad E[\lambda_t] < \infty. \]

The *local intensity function* $a : [0, T^*] \times \{0, 1, \ldots, n\} \mapsto \mathbb{R}_+$ at $t = 0$ is defined as

\[ a(t, i) := E^\mathbb{Q}[\lambda_t | N_{t-} = i, \mathcal{F}_0]. \tag{1} \]

If $\mathbb{Q}(N_{t-} = i | \mathcal{F}_0) = 0$, we set $a(t, i) = 0$ by convention. We call $\lambda^\text{eff}_t := a(t, N_{t-})$ the *effective intensity* of the loss process.
Mimicking marked point processes with Markovian jump processes

<table>
<thead>
<tr>
<th>Proposition 1 (Cont and Minca (2008))</th>
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</thead>
</table>

Consider any non-explosive jump process $(L_t)$ with an intensity $(\lambda_t)$ and i.i.d. jumps with distribution $G$. Define $(\tilde{L}_t)$ as the Markovian jump process with jump size distribution $G$ and intensity $(a(t, \tilde{N}_{t-}))$. Then, for any $t \in [0, T^*]$, $L_t$ and $\tilde{L}_t$ have the same distribution conditional on $\mathcal{F}_0$. In particular, the flow of marginal distributions of $(L_t)$ only depends on the intensity $(\lambda_t)$ through its conditional expectation $a(., .)$. 
Proposition 1 (Cont and Minca (2008))

Consider any non-explosive jump process \((L_t)\) with an intensity \((\lambda_t)\) and i.i.d. jumps with distribution \(G\). Define \((\tilde{L}_t)\) as the Markovian jump process with jump size distribution \(G\) and intensity \((a(t, \tilde{N}_t^-))\). Then, for any \(t \in [0, T^*]\), \(L_t\) and \(\tilde{L}_t\) have the same distribution conditional on \(\mathcal{F}_0\). In particular, the flow of marginal distributions of \((L_t)\) only depends on the intensity \((\lambda_t)\) through its conditional expectation \(a(\cdot, \cdot)\).

- The local intensity function is an analogue to the local volatility function

\[
(\sigma_{\text{local}}(t, K))^2 = E^Q[\sigma_t^2 | \mathcal{F}_0, S_t = K]
\]

for stochastic volatility models.

- Gyöngy (1986) shows a mimicking theorem for Itô processes.

- Bentata and Cont (2009) show a more general mimicking theorem for discontinuous semimartingales.
For a Markovian jump process, the transition probabilities \( Q(N_T = i | \mathcal{F}_0) = q(T, i) \) can be computed by solving a Fokker-Planck equation: for \( T \in (0, T^*] \),

\[
\begin{align*}
\partial_T q(T, 0) &= -a(T, 0)q(T, 0), \\
\partial_T q(T, i) &= -a(T, i)q(T, i) + a(T, i - 1)q(T, i - 1), \quad i = 1, \ldots, n - 1, \\
\partial_T q(T, n) &= a(T, n - 1)q(T, n - 1),
\end{align*}
\]

with initial condition \( q(0, 0) = 1, q(0, i) = 0 \) for \( i = 1, \ldots, n \).

- With the transition probabilities, we can compute the prices of index default swaps and CDO tranches.
Definition 2

Consider the equity tranche of a synthetic CDO with detachment point \( K \). The expected remaining notional value of this equity tranche at time \( T \) is equal to

\[
P(T, K) := E^\mathbb{Q}[ (K - L_T)^+ | \mathcal{F}_0].
\]

We follow the notation in Cont and Savescu (2008) and call this quantity the *expected tranche notional* with maturity \( T \) and strike \( K \).
Expected tranche notionals

Consider the equity tranche of a synthetic CDO with detachment point $K$. The expected remaining notional value of this equity tranche at time $T$ is equal to

$$P(T, K) := E^Q[(K - L_T)^+ | \mathcal{F}_0].$$

We follow the notation in Cont and Savescu (2008) and call this quantity the *expected tranche notional* with maturity $T$ and strike $K$.

The mark-to-market value of a CDO tranche $[a, b]$ with upfront payment $U^{[a,b]}$ and periodic spread $s^{[a,b]}$ is equal to:

$$MTM^{[a,b]} = U^{[a,b]}(b - a) + s^{[a,b]} \sum_{t_j > 0} D(0, t_j)(t_j - t_{j-1}) \left[ P(t_j, b) - P(t_j, a) \right]$$

$$- \sum_{j=1}^{m} D(0, t_j) \left[ P(t_j, a) - P(t_j, b) - P(t_{j-1}, a) + P(t_{j-1}, b) \right]$$

which is *linear* in the expected tranche notionals.
Expected tranche notionals

Property 1 (Static arbitrage constraints)

(a) $P(T, K) \geq 0$,
(b) $P(T, 0) = 0$,
(c) $P(0, K) = K$,
(d) $K \mapsto P(T, K)$ is convex,
(e) $P(T_2, K_1) - P(T_1, K_1) \geq P(T_2, K_2) - P(T_1, K_2)$ for any $T_1 \leq T_2$, $K_1 \leq K_2$,
(f) $K \mapsto P(T, K)$ is continuous and piecewise linear on $[(i - 1)\delta, i\delta]$, $i = 1, \ldots, n$.

All constraints are linear in the expected tranche notionals.
Cont and Savescu (2008) show that the expected tranche notionals can be computed directly from the local intensity function by solving a system of forward differential equations: for $T \in (0, T^*]$, $i = 1, \ldots, n$,

$$\partial_T P(T, i\delta) = -a(T, 0)P(T, \delta) - \sum_{k=1}^{i-1} a(T, k)\nabla_K^2 P(T, (k - 1)\delta)$$

with initial condition $P(0, i\delta) = i\delta$

where $\nabla_K$ is the forward difference operator in strike:

$$\nabla_K F(T, i\delta) := F(T, (i + 1)\delta) - F(T, i\delta)$$

for any function $F : [0, T^*] \times (i\delta)_{i=0,\ldots,n-1} \mapsto \mathbb{R}$. 


Roadmap

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Inversion formula

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Theorem 3 (Inversion formula)

Consider a portfolio loss process $L_t = \delta N_t$ where $(N_t)$ admits an intensity $(\lambda_t)$ and

$$\forall t \in (0, T^*], \quad E^Q[\lambda_t | \mathcal{F}_0] < \infty,$$

the local intensity function defined by (1) is given by

$$a(T, i) = \begin{cases} 
-\frac{\partial_T P(T, \delta)}{P(T, \delta)}, & i = 0, \\
-\nabla_K \partial_T P(T, i\delta) \frac{\nabla^2_K P(T, (i-1)\delta)}{\nabla^2_K P(T, (i-1)\delta)}, & i = 1, \ldots, n-1, \\
0, & i = n,
\end{cases}$$

for all $T \in (0, T^*]$, and $P(T, i\delta) = E^Q[(\delta i - L_T)^+ | \mathcal{F}_0]$. 


Theorem 4 (Local intensity implied by expected tranche notionals)

Let \( \{P(T, i\delta)\}_{T\in[0, T^*], i=0,...,n} \) be a (complete) set of expected tranche notionals verifying Property 1 and define the function \( a : (0, T^*] \times \{0, 1, .., n\} \) by

\[
a(T, i) = \begin{cases} 
-\frac{\partial_T P(T, \delta)}{P(T, \delta)}, & i = 0, \\
-\nabla_K \partial_T P(T, i\delta) \frac{\nabla^2_K P(T, (i-1)\delta)}{\nabla^2_K P(T, (i-1)\delta)}, & i = 1, ..., n-1, \\
0, & i = n,
\end{cases}
\]

for all \( T \in (0, T^*]. \) If \( a(., .) \) is bounded, there exists a Markovian point process \( (M_t) \) with intensity \( \gamma_t = a(t, M_{t-}) \) defined on some probability space \( (\Omega_0, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_0) \) such that

\[
\forall T \in [0, T^*], \quad \forall i \in \{0, ..., n\}, \quad P(T, i\delta) = E^{\mathbb{Q}_0}[(\delta i - \delta M_T)^+ | \mathcal{G}_0].
\]
The inversion formula is an analogue to the Dupire (1994) formula for diffusion models:

\[
\sigma^2(T, K) = \frac{2}{K^2} \frac{\partial_T C(T, K)}{\partial_K^2 C(T, K)}, \quad T \geq 0, K \geq 0
\]

where \(C(T, K)\) is the call price with maturity \(T\) and strike \(K\).
The inversion formula is an analogue to the Dupire (1994) formula for diffusion models:

\[
\sigma^2(T, K) = \frac{2}{K^2} \frac{\partial_T C(T, K)}{\partial^2_K C(T, K)}, \quad T \geq 0, K \geq 0
\]

where \( C(T, K) \) is the call price with maturity \( T \) and strike \( K \).

A similar formula, but expressed in terms of the marginal distribution, has been shown by Schönbucher (2005):

\[
a(T, i) = \frac{-\sum_{k=0}^{i} \partial_T Q(L_T = i\delta|\mathcal{F}_0)}{Q(L_T = i\delta|\mathcal{F}_0)}, \quad i = 0, \ldots, n-1, \quad T \in (0, T^*].
\]

However, expressing the value of CDO tranche in terms of marginal distribution is more difficult while it can be expressed in terms of a small set of expected tranche notionals.
Roadmap

- Credit portfolio loss models
- CDO market Data
  - Simulation
  - Quadratic programming

Expected tranche notionals
  - Inversion formula

Local intensity function
  - Model comparison
  - Hedging CDO tranches
  - Pricing exotic financial products
Recovery of expected tranche notionals

- Given a set of CDO tranche spreads, we want to recover expected tranche notionals \( \{ P(t_j, i\delta) \}_{j=1,...,m; i=1,...,n} \) which must satisfy:
  - Static arbitrage constraints
  - Mark-to-market value constraints
Given a set of CDO tranche spreads, we want to recover expected tranche notionals \( \{P(t_j, i\delta)\}_{j=1,\ldots,m; i=1,\ldots,n} \) which must satisfy:

- Static arbitrage constraints
- Mark-to-market value constraints

Both static arbitrage and the mark-to-market value constraints are linear in the expected tranche notionals.

Recovering the expected tranche notional can be achieved by solving a linear system of inequalities:

\[
\begin{align*}
\mathbf{A} \mathbf{p} &= \mathbf{b}, \quad \text{(Market CDO)} \\
\mathbf{B} \mathbf{p} &\leq \mathbf{e} \quad \text{(Static arbitrage)}
\end{align*}
\]

where \( \mathbf{p} \) is a vector of expected tranche notionals.
Given a set of CDO tranche spreads, we want to recover expected tranche notionals \( \{ P(t_j, i\delta) \}_{j=1,...,m; i=1,...,n} \) which must satisfy:

- Static arbitrage constraints
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Both static arbitrage and the mark-to-market value constraints are \textit{linear} in the expected tranche notionals.

Recovering the expected tranche notional can be achieved by solving a linear system of inequalities:

\[
A \, p = b, \quad \text{(Market CDO)} \\
B \, p \leq e, \quad \text{(Static arbitrage)}
\]

where \( p \) is a vector of expected tranche notionals.

However, the linear system may have infinitely many solutions.
In order to guarantee a unique solution, we solve the following convex optimization problem with linear constraints:

$$\min_{\mathbf{p}} \quad f(\mathbf{p})$$

s.t.\quad A \mathbf{p} = \mathbf{b} \quad \text{(Market CDO)}$$

$$\mathbf{B} \mathbf{p} \leq \mathbf{e} \quad \text{(Static arbitrage)}$$

where

$$f(\mathbf{p}) = \sum_{j=0}^{m} \sum_{i=1}^{n} w_{ij} \left( P(t_j, i\delta) - \tilde{P}(t_j, i\delta) \right)^2$$

where \((w_{ij})\) are weights, and \(\{\tilde{P}(t_j, i\delta)\}\) is a reference set of expected tranche notionals.

This is a quadratic programming problem.

The calibration algorithm is non-parametric.
Algorithm 1

1. Compute matrices $\mathbf{A}$ and $\mathbf{b}$ using observed CDO tranche spreads, and matrix $\mathbf{B}$ and $\mathbf{e}$ according to static arbitrage constraints.

2. Solve quadratic programming problem and obtain a set of arbitrage-free expected tranche notionals which is consistent with the CDO tranche spreads.

3. Convert the calibrated expected tranche notionals into local intensity function using formula in Theorem 2.
Credit portfolio loss models \quad \text{Simulation} \quad \text{Quadatic programming} \quad \text{Inversion formula} \quad \text{Local intensity function} \quad \text{Model comparison} \quad \text{Hedging CDO tranches} \quad \text{Pricing exotic financial products} \quad \text{CDO market Data}
Application to iTraxx IG data

- We apply our algorithm to iTraxx IG S9 data on 20 September 2006 and 25 March 2008.
- We also compare the results to
  (1) Parametric model by Herbertsson (2008),

<table>
<thead>
<tr>
<th>Tranche</th>
<th>0%-3%</th>
<th>3%-6%</th>
<th>6%-9%</th>
<th>9%-12%</th>
<th>12%-22%</th>
<th>22%-100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market bid</td>
<td>37.7%</td>
<td>441.6</td>
<td>270.2</td>
<td>174.4</td>
<td>97.4</td>
<td>42.8</td>
</tr>
<tr>
<td>Market ask</td>
<td>39.7%</td>
<td>466.6</td>
<td>290.2</td>
<td>189.4</td>
<td>110.7</td>
<td>46.9</td>
</tr>
<tr>
<td>QP</td>
<td>38.4%</td>
<td>451.9</td>
<td>279.0</td>
<td>181.1</td>
<td>103.2</td>
<td>44.3</td>
</tr>
<tr>
<td>Entropy</td>
<td>38.6%</td>
<td>453.3</td>
<td>279.5</td>
<td>181.2</td>
<td>103.4</td>
<td>44.6</td>
</tr>
<tr>
<td>Parametric</td>
<td>38.7%</td>
<td>454.1</td>
<td>280.2</td>
<td>181.9</td>
<td>104.1</td>
<td>44.8</td>
</tr>
</tbody>
</table>

Table 1: CDO tranche spreads of 5Y iTraxx Europe IG Series 9 on 25 March 2008. Quotes are given in bps except for equity tranches which are quoted as upfront in percent with 500bps periodic coupons.

- All calibrated spreads are well-within bid-ask.
Local intensity function

![Graphs showing local intensity functions based on different calibration approaches.](image)

**Figure 2:** Local intensity functions based on different calibration approaches. Data: 5Y iTraxx Europe IG S9 on 20 September 2006 (top) and 25 March 2008 (bottom).

- Different calibration methods yield significantly different local intensity functions.
- For each method, the local intensity functions are similar for different dataset.
Stability analysis

To examine the stability of the calibration methods, we apply a 1% proportional shift to all CDO market spreads, recalibrate the local intensity function to the shifted CDO spreads and measure the magnitude of the changes using the Frobenius norm:

$$\left( \sum_{i=0}^{n} \sum_{j=0}^{q} |a(T_j, i) - \hat{a}(T_j, i)|^2 \right)^{1/2}$$

where \( \{a(T_j, i)\} \) and \( \{\hat{a}(T_j, i)\} \) are, respectively, the local intensity functions calibrated to the original and perturbed CDO tranche spreads.

<table>
<thead>
<tr>
<th>Date</th>
<th>QP</th>
<th>Parametric</th>
<th>Entropy Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>20-Sep-06</td>
<td>56.2</td>
<td>32116.2</td>
<td>(2.0 \times 10^{-2})</td>
</tr>
<tr>
<td>25-Mar-08</td>
<td>673.2</td>
<td>728.3</td>
<td>(2.0 \times 10^{-1})</td>
</tr>
</tbody>
</table>

Table 2: Frobenius norm of the changes in the local intensity function with respect to 1% proportional increase in the CDO spreads. Data: 5Y iTraxx Europe IG S6 on 20 September 2006 and S9 on 25 March 2008.

- Non-parametric methods are more stable than the parametric method.
- Similar findings in studies using equity derivatives: Cont and Tankov (2004).
Forward starting tranche spreads

A forward tranche with attachment-detachment interval \([a, b]\) can be valued as the forward value of a tranche with adjusted interval \([a', b']\) where \(a' = \min(1, a + L_t)\) and \(b' = \min(1, b + L_t)\). This dependence of the payoff on the loss makes the forward tranche path dependent.

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<tr>
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</thead>
<tbody>
<tr>
<td></td>
<td>QP</td>
<td>Parametric</td>
</tr>
<tr>
<td>0% - 3%</td>
<td>12.05</td>
<td>12.25</td>
</tr>
<tr>
<td>3% - 6%</td>
<td>2.72</td>
<td>17.89</td>
</tr>
<tr>
<td>6% - 9%</td>
<td>2.46</td>
<td>3.18</td>
</tr>
<tr>
<td>9% - 12%</td>
<td>2.21</td>
<td>0.79</td>
</tr>
<tr>
<td>12% - 22%</td>
<td>1.59</td>
<td>0.36</td>
</tr>
<tr>
<td>22% - 100%</td>
<td>0.03</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 3: Spreads of forward starting tranches which start in 1 year and mature 3 years afterwards. Data: 5Y iTraxx Europe IG S6 on 20 September 2006 and S9 on 25 March 2008.

- Forward tranche spreads can be different by more than double, even the local intensity functions are calibrated to the same market CDO spreads ⇒ Substantial model uncertainty
Hedge ratios

In the local intensity framework, the market is complete and the self-financing strategy to replicate the payoff of a CDO tranche involves trading the underlying index default swap. The corresponding hedge ratio, which is known as the *jump-to-default ratio*, is defined by:

\[
\frac{\nu^{[a,b]}(t, N_t + 1) - \nu^{[a,b]}(t, N_t)}{\nu^{\text{index}}(t, N_t + 1) - \nu^{\text{index}}(t, N_t)}
\]

where \(\nu(t, m)\) denotes the mark-to-market value conditional on \(m\) defaults being occurred by time \(t\).

<table>
<thead>
<tr>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QP Parametric</td>
<td>Entropy Min</td>
</tr>
<tr>
<td>0% - 3%</td>
<td>6.29</td>
<td>20.97</td>
</tr>
<tr>
<td>3% - 6%</td>
<td>2.12</td>
<td>5.16</td>
</tr>
<tr>
<td>6% - 9%</td>
<td>1.63</td>
<td>2.00</td>
</tr>
<tr>
<td>9% - 12%</td>
<td>1.52</td>
<td>1.02</td>
</tr>
<tr>
<td>12% - 22%</td>
<td>1.47</td>
<td>0.48</td>
</tr>
<tr>
<td>22% - 100%</td>
<td>0.67</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Table 4: Jump-to-default ratios computed from the calibrated local intensity functions. Data: 5Y iTraxx Europe IG S6 on 20 September 2006 and S9 on 25 March 2008.

- Jump-to-default ratios are also significantly different across calibration methods ⇒ Substantial model uncertainty
We compare the local intensity functions of six different models:

1. **Parametric local intensity model**: Herbertsson (2008)
   - $\lambda_t = (n - N_t-) \sum_{k=0}^{N_t-} b_k$
   - $\lambda_{t}^{\text{eff}} = \lambda_t$
We compare the local intensity functions of six different models:

1. **Parametric local intensity model**: Herbertsson (2008)
   \[
   \lambda_t = (n - N_t -) \sum_{k=0}^{N_t} b_k \\
   \lambda_{t}^{\text{eff}} = \lambda_t
   \]

2. **Bivariate spread-loss model**: Arnsdorf and Halperin (2008)
   \[
   \lambda_t = e^{X_t} (n - N_t -) \sum_{k=0}^{N_t} b_k \\
   \text{where } dX_t = \kappa (b - X_t) dt + \sigma dW_t
   \]
Comparison of credit portfolio loss models

We compare the local intensity functions of six different models:

1. **Parametric local intensity model**: Herbertsson (2008)
   
   \[ \lambda_t = (n - N_t^-) \sum_{k=0}^{N_t^-} b_k \]
   
   \[ \lambda_{\text{eff}} = \lambda_t \]

2. **Bivariate spread-loss model**: Arnsdorf and Halperin (2008)
   
   \[ \lambda_t = e^{X_t} (n - N_t^-) \sum_{k=0}^{N_t^-} b_k \]
   
   where \( dX_t = \kappa (b - X_t) dt + \sigma dW_t \)

3. **Shot-noise model**: Gaspar and Schmidt (2008)
   
   \[ \lambda_t = \eta_t + J_t \]
   
   where \((\eta_t)\) is a CIR process and \((J_t)\) is a compound Poisson process with exponential jump size.
   
   A semi-analytical expression for the local intensity function:

   \[
   a(T, k) = \left. \frac{\partial^k}{\partial \theta^k} \right|_{\theta=-1} \frac{\partial}{\partial T} \frac{1}{\theta} S(\theta, T) \\
   \left. \frac{\partial^k}{\partial \theta^k} \right|_{\theta=-1} S(\theta, T)
   \]

   where \( S(\theta, T) \) is the Laplace transform of the cumulative portfolio default intensity.
4. **Gaussian copula model**: Li (2000)

Given a family of marginal default time distributions \((F_i, i = 1, \ldots, n)\), the joint distribution of the default times \(\tau_i\) is modeled by first defining latent factors \(X_i = \rho Z_0 + \sqrt{1 - \rho^2} Z_i\), where \(Z_0, Z_i\) are i.i.d. standard normal random variables. Defining the default times by

\[
\tau_i = F_i^{-1}(F_{X_i}(X_i)),
\]

where \(F_{X_i}(\cdot)\) denotes the distribution of \(X_i\).
4. **Gaussian copula model**: Li (2000)
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     \[
     \tau_i = F_i^{-1}(F_{X_i}(X_i)),
     \]
   - where \( F_{X_i}(.) \) denotes the distribution of \( X_i \).

5. **Student-t copula model**: Demarta and McNeil (2005)
   - Same as the Gaussian copula case but replacing normal latent factors by \( X_i = \sqrt{\nu/V} \left( \rho Z_0 + \sqrt{1 - \rho^2} Z_i \right) \) where \( V \sim \chi_\nu^2 \).
4. **Gaussian copula model**: Li (2000)
   - Given a family of marginal default time distributions \( (F_i, i = 1, \ldots, n) \),
     the joint distribution of the default times \( \tau_i \) is modeled by first defining latent factors
     \( X_i = \rho Z_0 + \sqrt{1 - \rho^2} Z_i \), where \( Z_0, Z_i \) are i.i.d. standard normal random variables. Defining the default times by
     \[
     \tau_i = F_i^{-1}(F_{X_i}(X_i)),
     \]
     where \( F_{X_i}(.) \) denotes the distribution of \( X_i \).

5. **Student-t copula model**: Demarta and McNeil (2005)
   - Same as the Gaussian copula case but replacing normal latent factors by
     \( X_i = \sqrt{\nu/V} \left( \rho Z_0 + \sqrt{1 - \rho^2} Z_i \right) \)
     where \( V \sim \chi^2_\nu \).

   - The default intensity for obligor \( i \) follows:
     \[
     \lambda_t^i = X_t^i + a_i X_t^0
     \]
     where
     \[
     dX_t^i = \kappa_i(b_i - X_t^i)dt + \sigma_i \sqrt{X_t^i} dW_t^i + dJ_t^i
     \]
Local intensity functions implied by credit portfolio loss models

![Graphs of local intensity functions for different models: Herbertsson model, Bivariate spread-loss model, Shot-noise model, Gaussian copula model, Student-t copula model, Affine jump-diffusion model.](image)

**Figure 3:** Local intensity functions implied by credit portfolio loss models. Data: 5Y iTraxx Europe IG S9 on 25 March 2008.

- Static copula models have similar effective intensities as the dynamic affine jump-diffusion model
  ⇒ Market prices alone are insufficient to discriminate between these model classes.
We derive an inversion formula for the local intensity function which is an analogue to the Dupire (1994) local volatility function.

Inversion formula + QP ⇒ a simple, efficient and stable calibration algorithm for the effective default intensity.

Even under the same modeling framework, there are substantially differences in model-dependent quantities such as jump-to-default ratios and forward tranche prices.

⇒ Model uncertainty

We observe similar local intensity functions implied by models defined in different manners, e.g. static copula models vs dynamic affine jump-diffusion model.

⇒ Market prices alone are insufficient to discriminate between these model classes.